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REPORT No. R.187

AN INTRODUCTION TO THE ENGINE RESPONSE RATE PROBLEM

PART I. CONTROL SYSTEM THEORY RELEVANT TO THE ENGINE PROBLEM

by

H. TAYLOR

MARCH, 1956

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An introduction to the engine response rate problem

PART I

Control system theory relevant to the engine problem

- by -

H. Taylor

SUMMARY

As the flight conditions of gas turbine engined aircraft become more severe, engine control system complexity will be increased and the performance demanded of the control system in respect of the number of parameters to be controlled, the range of control, and the rate of response and sensitivity, will become more difficult to achieve.

An increasing effort is accordingly being devoted to control system design problems and to consideration of the power plant comprising the engine and its variable geometry, together with the control system, as a complete and integrated unit.

Since the engine element itself is a dynamic system, the response of which is a function of basic engine performance and operating conditions, the study of engine response behaviour under controlled conditions must result in the theory and technique of control system and engine performance becoming closely related.

In Part I of this paper an introduction is given to the fundamentals of control system theory relevant to the problem; this is intended as a guide to the non-specialist rather than as a treatise on servo-systems.

The theory and technique of servo-system design are highly developed and provided the engine problem can be rationalised by acceptable means, the wealth of mathematical technique already available should enable the various problems to be resolved by established methods.

In Part II of the paper (R.188) the engine problem is considered and an analysis made of current literature, much of it work undertaken in the U.S.A. by the N.A.C.A.

It has been established that the response of the engine element is basically that of a simple exponential lag system and that within certain limits the engine time constant (which completely defines such a system) may be calculated from engine component performance data. The effect of

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altitude on engine time constant may also be calculated and is found to be considerable. It has been shown however, that although this data enables a preliminary assessment to be made, secondary effects including additional lag terms, which may predominate in importance in relation to high gain fast response systems are not included, and the response under high altitude conditions, when significant changes occur in component performance, has yet to be investigated.

The engine experiments required, thus relate to investigating the departure from simple theory which occur in the engine element and the reasons for them; this data is obviously required for a range of engines. In addition, data is required which will enable a similar assessment to be made over as wide a range of altitude and forward speeds as possible.

Consideration of the methods for calculating engine response shows a need for simplified methods which can be applied at a very early stage in future engine development, since until a suitable engine test facility becomes available, the development testing of control systems for the supersonic high altitude case would appear to be very difficult to simulate.

Two basic methods are available for response tests, namely step change or harmonic input functions, and it has been shown that an arbitrary input may be substituted for the step changes at the expense of computing effort.

Step change or arbitrary inputs are simple to apply in practice, but the information obtained is mainly limited to time constant and gain data; further analysis may extend these results to yield the frequency response and secondary effect properties of the system, but the method is sensitive to experimental error. For the development of high gain fast response systems, there would appear to be no alternative to the more complex harmonic response method. Considerable care is necessary in performing these tests to achieve a true harmonic input and the general experimental methods used are extended to their limit in respect of accuracy, especially in relation to small engines.

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ILLUSTRATIONS (cont'd)

1.0 Introduction

The current N.G.T.E. programme of work, which is considered in Part II, is concerned primarily with the response of the basic engine element to controlled changes in fuel flow, but this must be undertaken with full consciousness of the part played by this element in the overall system; the form which the programme follows is dictated by the use to which the results are to be put in control system design. As knowledge is gained the work will become progressively more linked with servo-system design and with the overall system problems of sensitivity and stability.

The theory and development of control devices made considerable strides during World War II, but for security reasons the work was not published until after the war. A wealth of data and specialised technique was released eventually, which due to the intensive development which had occurred, resulted in the subject tending to be incomprehensible except to the specialist.

Since the engine problem is only partly one of servo-systems, being largely a performance study, it embraces a range of interests many of which are unconnected with servo-system design. Whilst there are excellent books and papers dealing with controls and servo-systems, there appears to be a need for an elementary paper restricted to aspects of the subject relevant to the gas turbine response rate programme.

Unfortunately it is necessary to carry this further than justified by the present state of the response rate problem due to the style of reporting employed in current reports. In the case of many American papers, established but advanced techniques are applied to problems which are in fact capable of presentation in a more elementary and as far as the non-specialist is concerned, more intelligible manner. For this reason techniques developed for control system synthesis are discussed, even though the subject in general is outside the scope of this paper.

For more information on control systems, works such as Theory of Control by Macmillan, and Principles of Servo-mechanisms by Brown and Campbell (References 1 and 2) may be consulted; both also contain excellent bibliographies of the literature relevant to control systems, as does Reference 3.

Harmonic response has been introduced in this paper via a step function approach, since the latter has a readily understood practical meaning in relation to gas turbine operation. This is not however intended to imply that the step function approach is of greater significance and consideration of servo-system design, where stability is a major factor, would probably favour an initial approach via the harmonic response properties of the system.

Simple control systems have been used for many years to perform routine tasks and to control remotely. Controls for regulation of engine speed and industrial processes have been in widespread use for a long time and over a number of years the theory of control systems has been developed independently by a number of groups. This has resulted in common problems being encountered in electrical engineering, communications engineering, process control engineering etc. and being resolved independently; inevitably different terminology was applied in the different fields and interchange of technique rendered impossible. As a result of the late war many of these fields were brought together and a common approach and terminology employed. A considerable number of

text books and papers on the subject have been published in the U.S.A. and these have tended to place emphasis on the American approach and techniques; this is especially true in respect of the Laplace transform method which is universally applied in the U.S.A. to simple and complex problems alike. Whilst it enables discontinuous functions to be dealt with without difficulty and is a powerful advantage in dealing with high order systems and for synthesis, it is not greatly superior to the classical operator method for the analysis of simple systems. Care must also be exercised in the solution of transient problems by the Laplace transform method to ensure that the "initial condition" terms are correctly treated. Nevertheless much of the published literature makes exclusive use of this method and it must be considered in some detail.

Much of the theory, especially that related to harmonic input signals, is closely related to vibration theory, the various control system loops being analogous to the elements of a complex vibratory system, the stability and response behaviour of both being identical. An important feature of control system work lies in the importance of random and discontinuous input functions and in the development of control system theory as a tool for the synthesis, i.e. development and improvement of particular complex systems, as opposed to mere analysis of system behaviour.

2.0 Open chain and closed loop control systems

Controls divide readily into open chain and closed loop systems. In the open chain system the control operates to provide an output according to a predetermined schedule or calibration and is unaffected by the resulting output or changes in output. In a closed loop system the control is set to give the desired output and the output is compared, preferably continuously, with the input. In this way control action is taken to bring the output to the desired value and to counteract any tendency to disturb it from this value.

A rheostat controlled hot-plate is an example of an open chain system; the rheostat may be calibrated in terms of hot-plate temperature and the desired value set accordingly. Any change in calibration, drop in voltage etc. cannot be accounted for, i.e. there is no feedback from the output for comparison with the command. This system is illustrated as a block diagram in Figure 1.

If we now measure the hot-plate temperature and compare it with a standard reference temperature controlled by the command, we can obtain a difference signal of appropriate sign indicating whether the temperature of the hot-plate is high or low; this signal may then be amplified and made suitable for operating a motor to adjust the rheostat and so bring the hot-plate temperature to the desired setting. Any change in condition will result in the generation of an error signal which will produce a corrective action; we thus have a closed loop system (Figure 1).

3.0 Regulators and servo-systems

The terms regulator and servo-system often lead to confusion and there is probably no universally agreed definition or distinction between the two. It may be said however that the regulator is in general of lower performance and is mainly concerned with maintaining the controlled quantity at a set level, e.g. the conventional voltage regulator, or engine top speed governor. The range of control is usually limited in respect of external disturbances and the error signal may generate the output function only; this will usually result in a steady state error when a large variation in input quantity is applied, or on application of external load. The latter is usually known as "load droop."

The servo-system however will normally control over a wide range of input functions and will control with random input or load variation. The system is usually power amplifying and the error signal will normally generate rate of change of output in addition to output, thus enabling the device to exhibit a "follow up" characteristic so that there is negligible steady state error between output and input. This is probably the main difference between the regulator and servo-system but must not be taken as implying that the regulator is of necessity an inferior device, since in many applications it gives excellent service and exhibits a high order of accuracy.

Finally it must be said that many practical systems are not capable of precise identification by such hard and fast rules as the above.

There is less ambiguity in the definition of a servo-mechanism, which is often defined as a servo-system with a mainly mechanical output.

4.0 Basic control systems and their behaviour

4.1 Class "0" system - displacement error system or simple proportional controller

4.1.1 Open chain system

Consider a positioning device comprising a long shaft free from inertia and friction, but possessing elasticity in such a way that it has a stiffness k in torque per unit angular twist units. Also consider the shaft subject to a load torque Γ_L at the output end (Figure 2). The symbols used are identified in Appendix II.

Then if θ_i is the input angle, θ_i is the angle to which the output would set in an ideal system; let the actual output angle be θ_o

$$k\,(\theta_i - \theta_o) = \Gamma_L$$

$$\theta_i - \theta_o = \frac{\Gamma_L}{k} = \theta \qquad \ldots \qquad \ldots \qquad (1)$$

where θ is the error angle in output setting; in some control system work the opposite sign convention is used, which would make

$\frac{\Gamma_L}{k}$ negative for the case stated.

Thus in the presence of a load torque there is always an error in the output.

4.1.2 Closed loop system

Consider a similar system to the above but with the addition of the feedback loop and differential device providing the error signal θ and the amplifier of amplification factor or "gain" μ which is necessitated by the small magnitude of the error signal θ.

$$k\,(\mu\theta - \theta_o) = \Gamma_L$$

By definition:- $$\theta_i - \theta_o = \theta$$

$$\mu\theta_i - \mu\theta_o - \theta_o = \frac{\Gamma_L}{k}$$

$$\theta_o = \frac{\mu}{1 + \mu} \cdot \theta_i - \frac{1}{1 + \mu} \cdot \frac{\Gamma_L}{k}$$

$$\theta = \frac{1}{1 + \mu} \cdot \theta_i + \frac{1}{1 + \mu} \cdot \frac{\Gamma_L}{k} \quad \ldots \quad \ldots \; (2)$$

Comparison of Equations (1) and (2) at the set point when $\theta_i = o$ indicates that with the closed loop system the error has been reduced from $\frac{\Gamma_L}{k}$ for the open chain to $\frac{1}{1 + \mu} \cdot \frac{\Gamma_L}{k}$, i.e. for an amplifier gain of 10 it is reduced to one eleventh of the former value.

It should be noted that in the absence of a load torque the closed loop system exhibits an error when away from the set point of $\frac{1}{1 + \mu} \cdot \theta_i$, which would class this system as a regulator rather than a servo-mechanism.

The increased lagging error with increase in load torque is an example of load droop and is a common characteristic of regulators.

The close relation to feedback amplifiers will be noted from the foregoing and it is not surprising that much of the knowledge and technique employed in control system work was developed in the course of feedback amplifier design.

4.2 Class "1" system - zero displacement error system

Whilst consideration of the Class "0" system has emphasised a number of important points in control system performance, neglect of friction and storage terms has reduced the discussion to displacement and load parameters without the conception of time functions, since movement of the input produced instantaneous movement of the output.

Consideration is now given to the closed loop system discussed previously, but with the substitution of rate of change of output proportional to error rather than output proportional to error (Figure 3).

The conception of time is now introduced by $\theta_i(t)$, $\theta_o(t)$, $\theta(t)$, and $\Gamma_L(t)$ as time functions.

$$\theta(t) = \theta_i(t) - \theta_o(t)$$

$$k\left[K\int \theta(t)dt - \theta_o(t)\right] = \Gamma_L(t)$$

$$k\left[K\,\theta(t) - \frac{d\,\theta_o(t)}{dt}\right] = \frac{d\,\Gamma_L(t)}{dt}$$

$$\left.\begin{aligned} k\,K\,\theta_o(t) + k\frac{d\,\theta_o(t)}{dt} &= k\,K\,\theta_i(t) - \frac{d\,\Gamma_L(t)}{dt} \\ k\,K\,\theta(t) + k\frac{d\,\theta(t)}{dt} &= k\frac{d\,\theta_i(t)}{dt} + \frac{d\,\Gamma_L(t)}{dt} \end{aligned}\right\}$$

Following a step change of θ_i, when $t \rightarrow \infty$ and derivatives are zero:-

$$\theta_{oss} = \theta_i$$

and $$\theta = 0$$

irrespective of the value of θ_i and Γ_L, where θ_{oss} denotes the eventual steady state value of θ_o. The integral term has thus provided a system with zero steady state error and load droop for a step input of displacement, i.e. a servo-system and also in the example quoted a servo-mechanism.

In the absence of load torque:-

$$K\,\theta_o(t) + \frac{d\,\theta_o(t)}{dt} = K\,\theta_i(t) \quad \dots \quad \dots \quad \dots \quad (3)$$

Writing the operator $$D \equiv \frac{d}{dt}$$

$$(D + K)\,\theta_o(t) = K\,\theta_i(t) \quad \dots \quad \dots \quad \dots \quad (4)$$

$$D\,\theta_o(t) = K\,\theta(t)$$

If a step input of θ_i is applied:-

$$\frac{d\,\theta_o(t)}{\theta_i - \theta_o(t)} = K\,dt$$

$$\therefore \quad \theta_i - \theta_o(t) = e^{c} \cdot e^{-Kt}$$ where c is an arbitrary constant.

When $t = 0$

$$\theta_o = 0$$

$$e^c = \theta_i$$

$$\frac{\theta_o(t)}{\theta_i} = \left[1 - e^{-Kt}\right]$$

It will be noted that K has the dimensions $\dfrac{\text{angle/time}}{\text{angle}} = \dfrac{1}{\text{unit of time}}$.

If we write $\tau = \frac{1}{K}$, τ has the unit of time and is known as the "time constant", to which considerable significance is attached.

Thus:-

$$\frac{\theta_o(t)}{\theta_i} = \left[1 - e^{-\frac{t}{\tau}}\right]$$

which represents a simple exponential lag response of the form shown in Figure 4.

When $t = \tau$

$$\frac{\theta_o}{\theta_i} = 0.632$$

When $t = 4\tau$

$$\frac{\theta_o}{\theta_i} = 0.982$$

which gives a guide to the response time for virtual completion of the output motion transient.

Initially when a step change of θ_i is applied:-

$$\theta = \theta_i \quad \text{since the output has not moved}$$

and from Equation (3):-

$$\theta_i = \tau \frac{d\,\theta_o}{dt}$$

Thus the tangent to the output response - time curve at $t = 0$ intersects the $\frac{\theta_o}{\theta_i} = 1$ line at $t = \tau$ (Figure 4).

We consider now the case of a step change in input velocity i.e. in $\frac{d\,\theta_i}{dt}$.

Equation (3) may be differentiated to give:-

$$\frac{d\,\theta_o(t)}{dt} + \tau\frac{d^2\theta_o(t)}{dt^2} = \frac{d\,\theta_i(t)}{dt}$$

Writing

$$\frac{d\,\theta_o(t)}{dt} = \omega_o(t)$$

and

$$\frac{d\,\theta_i(t)}{dt} = \omega_i(t)$$

For a step change in input velocity ω_i :-

$$\omega_i - \omega_o(t) = e^c e^{-\frac{t}{\tau}}$$

When $t = 0$

$$\theta_o = 0$$

$$\theta_i = 0$$

$\therefore$ From Equation (3) $\omega_o = 0$

and $\omega_i = e^c$

Thus $$\omega_o(t) = \omega_i\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\theta_o(t) = \omega_i\left(t + \tau e^{-\frac{t}{\tau}}\right) + c_1$$

When $$t = 0$$

$$\theta_o = 0$$

$$\therefore \quad c_1 = -\omega_i\tau$$

Hence $$\theta_o(t) = \omega_i t - \omega_i\tau\left(1 - e^{-\frac{t}{\tau}}\right)$$

In the steady state for large values of t:-

$$\theta_{oss} = \omega_i t - \omega_i\tau$$

The response to this step input of velocity is shown in Figure 4, where it can be seen that after the initial transient condition expressed by the term $\omega_i\tau \, . \, e^{-\frac{t}{\tau}}$ the output rotates in unison with the input with a lag error of $\omega_i\tau$. This may be minimised by keeping τ to a low value. The term $\omega_i\tau$ is often known by the misleading term "velocity lag" when in fact it is an angular displacement lag resulting from the velocity input. It will also be noted that when the input is arrested the output "follows up" and cancels the lag error and also that this lag is present without there being inherent lag in the controller, i.e. the integration has had the effect of adding a storage unit to the system.

It is clear from a study of the Class "0" and Class "1" systems that integration within the loop is necessary to remove steady state error and it is of interest to consider a double integration within the controller.

4.3 Class "2" system - zero velocity error system

This is shown in Figure 3.

$$k\left[K\iint\theta(t)\, dt \, . \, dt - \theta_o(t)\right] = \Gamma_L(t)$$

$$k\left[K\,\theta(t) - \frac{d^2\theta_o(t)}{dt^2}\right] = \frac{d^2\,\Gamma_L(t)}{dt^2}$$

$$k\,K\,\theta_o(t) + k\,\frac{d^2\theta_o(t)}{dt^2} = k\,K\,\theta_i(t) - \frac{d^2\,\Gamma_L(t)}{dt^2}$$

With zero load torque:-

$$(D^2 + K)\,\theta_o(t) = K\,\theta_i(t) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Hence $\theta_o(t) = \theta_i(1 - \cos\sqrt{K}\,t)$ ... ... (6)

for a step input of θ_i.

This represents a continuous oscillation and the system thus requires some form of damping for use in a practical case; it may be noted in passing that the simple pendulum is an example of a Class "2" system.

4.4 Order of control

It can be seen from the foregoing that in the absence of time lags, friction etc. controls may be grouped according to the number of integrations within the loop and will exhibit distinctive differential equations relating input, output and error; these are given in Table I for displacement-displacement systems and it can be seen that increase in the order of control tends to improve the steady state behaviour but tends to diminish the system stability. It is also clear that satisfactory operation of a control system necessitates a careful theoretical assessment of the arrangement if the accuracy, rate of response and inherent stability are to be acceptable. It should be noted that there is a clear distinction between order of control and the order of the differential equation, since these are not necessarily synonymous. This will be considered further in Section 4.5.1.

4.5 Consideration of a practical system

The simple systems discussed previously are unlikely to occur in practice when time lags, friction and inertia effects are present. These systems would in any event be little use in practice since there is no power amplification and the input and output cannot be physically separated.

4.5.1 Simple system with and without time lag

The positioning system depicted in Figure 5 illustrates a basic Ward Leonard set employed as a position control. Friction and inertia are neglected, but otherwise the system has the basic requirement of power amplification and the ability to have the input and output located remotely. A rotary potentiometer connected to the output shaft provides a voltage proportional to output angle which is compared with the voltage provided from the command rotary potentiometer measuring input angle: the resulting error signal is fed to an amplifier whose output controls the field current of the power generator, which is in turn driven by a constant speed motor. The servo-motor is fed with a constant field current and thus has an input voltage proportional to error.

Hence the voltage applied to the servo-motor $\propto \theta(t)$

Speed of the servo-motor $\propto$ input voltage $\propto \theta(t)$

Hence
$$\frac{d\,\theta_o(t)}{dt} = K\,\theta(t)$$

$$= K\left[\theta_i(t) - \theta_o(t)\right]$$

$$(D + K)\,\theta_o(t) = K\,\theta_i(t)$$

$$D\,\theta_o(t) = K\,\theta(t)$$

This may be compared with Equation (4) and represents a first order system.

Consider now the system depicted in Figure 5 in which the inductance and resistance of the generator field are introduced, i.e. the lag of the field winding:-

$$e\ (t) \;=\; L\frac{di_f(t)}{dt} + R\ i_f(t) \;=\; \mu\ \theta(t)$$

$$i_f\ (t) \;=\; \frac{K_1}{1 + \tau D}\ \theta(t) \qquad \ldots \quad \ldots \quad \ldots\ (7)$$

Where $K_1 = \dfrac{\mu}{R}$

$\tau = \dfrac{L}{R}$

We thus have a simple exponential lag interposed between the error and the output.

$$\text{Hence } D\ \theta_o(t) \;=\; \frac{K_2}{1 + \tau D}\ \theta(t) \qquad \ldots \quad \ldots \quad \ldots\ (8)$$

$$\text{and} \quad \theta_o(t) \;=\; \frac{K_2\ \theta_i(t)}{(\tau D^2 + D + K_2)}$$

Where $K_2 = K'\ K_1$

and $K'\ \mu\ \theta(t) = D\ (LD + R)\ \theta_o(t)$

For a step input displacement θ_i:-

$$\theta_o = \theta_i$$

$$\therefore\ \theta = 0$$

For a step input velocity ω_i:-

$$\theta_{ss} = \frac{\omega_i}{K_2}$$

Thus the addition of the time lag has not altered the basic first order response characteristic of zero displacement error for step input of displacement and displacement lag $\frac{\omega_i}{K}$ for a step input of velocity.

The response however is no longer a simple exponential for the step input case, and has oscillatory tendencies governed by the solution of the characteristic equation $(\tau D^2 + D + K_2)\ \theta_o(t) = 0$ and the value of τ and K_2 (Figure 6). This may be compared with Figure 4 for the basic system.

Reference to Equation (8) shows that the response is defined by:-

$$\theta_o(t) \;=\; \frac{K_2}{D\ (1 + \tau D)}\ \theta(t)$$

The governing equation with a second simple exponential delay would take the form:-

$$\theta_o(t) = \frac{K_2}{D\,(1 + \tau_1 D)\,(1 + \tau_2 D)}\,\theta(t)$$

It will be seen later that where the lag involves an inertia and spring force, or inductance and capacitance, the governing equation would have the form:-

$$\theta_o(t) = \frac{K_2}{D\,(1 + 2\zeta\tau D + \tau^2 D^2)}\,\theta(t)$$

which is described as a first order system with a complex exponential delay, the latter term following from the nature of the roots of the characteristic equation.

In general, lag terms have a de-stabilising effect and it will be noted that the order of the differential equation is of course raised by them.

It is appropriate at this juncture to consider also the finite delay or time lag caused, for example, by lost motion in a system, or by the reaction time of a human operator.

In this case the equation for the delay "element" is:-

$$\theta_o(t) = \theta_i(t - T)$$

where T denotes the time delay.

Or $$\theta_o(t) = \left[1 - TD + \frac{(TD)^2}{2!} - \frac{(TD)^3}{3!} \cdots\cdots\right]\theta_i(t)$$

$$= e^{-TD}\,\theta_i(t)$$

$$\therefore\ \theta_o(t) = \frac{\theta_i(t)}{e^{TD}}$$

The finite delay is thus equivalent to an exponential operator and introduces in effect an infinite order differential equation; it follows that the finite delay may have a severe de-stabilising effect. This is a most important factor in a practical design where such delays are frequently present. The effect may be more readily appreciated in relation to the harmonic response locus and is considered again in 12.0.

4.5.2 System with viscous friction and inertia

The systems discussed previously are unlikely to occur in practice without the presence of friction and inertia effects. A more practical remote position control (r.p.c.) will now be considered, similar to the last case in that the input and output may be spaced as desired and considerable output power controlled by a negligible input power. This system is depicted in Figure 5 and will be seen to comprise a constant armature current motor driving the output to which a rotary potentiometer is fitted measuring output angle; the voltage representing this angle is compared with the voltage equivalent of the input

angle (as set by the input potentiometer) and the resulting error signal fed to an amplifier driving the split field winding of the driving motor.

We assume that:-

Motor torque $\Gamma_M(t) \propto$ field current $i_f(t) \propto \theta(t)$

For viscous friction, friction torque $\propto \dfrac{d\ \theta_o(t)}{dt}$

Torque absorbed = torque applied

$$J \frac{d^2 \theta_o(t)}{dt^2} + f \frac{d\ \theta_o(t)}{dt} + \Gamma_L(t) = K\ \theta(t)$$

Where f is the viscous friction coefficient

$$\theta(t) = \frac{(J\ D^2 + f\ D)\ \theta_i(t) + \Gamma_L(t)}{(J\ D^2 + f\ D + K)} \quad \ldots \quad (9)$$

$$\theta_o(t) = \frac{K\ \theta_i(t) - \Gamma_L(t)}{(J\ D^2 + f\ D + K)} \quad \ldots \quad (10)$$

$$K\theta(t) = D\ (J\ D + f)\ \theta_o(t) + \Gamma_L(t) \quad \ldots \quad (11)$$

By putting D and powers of D equal to zero in Equation (9) for the steady state:-

$$\theta_{ss} = \frac{\Gamma_L}{K}$$

for a step input θ_i and constant load torque Γ_L.

It should be noted that the system is a servo-mechanism in that there is no error due to input disturbance; the effect of the output load to produce droop is a characteristic of all systems in which control torque is proportional to error only.

For a step input of velocity ω_i and constant load torque:-

$$\theta_{ss} = \frac{f\ \omega_i}{K} + \frac{\Gamma_L}{K}$$

If $J = 0$ we obtain the simple first order system, and if we put $f = 0$ the basic second order system which results in continuous oscillation of the output. Clearly at intermediate values of f and dependent upon the values of J and K there are output characteristics with varying oscillatory tendencies, which may be poorly damped but are not of course divergent. When $f = 0$ the system oscillates continously at a mathematically bounded finite value and is said to be neutrally stable.

Examination of Equation (11) and comparison with Equation (8) shows that the system is basically a first order control with a simple lag added. The differential equation however is of the second order.

The exact nature of the solution can be found by solution of the differential Equations (9) and (10).

The solution of a differential equation comprises the particular integral representing the forced or sustained solution added to the complementary function representing the transient or unforced system response. In the classical method of solution a forcing function must be established and the particular integral determined; with step inputs the forced or sustained solution must be the steady state response.

Hence:-

Complete solution = complementary function + steady state solution

The complementary function is obtained by setting the forcing function at zero, giving as the characteristic equation for Equation (9) or (10):-

$$(J\,D^2 + f\,D + K)\,\theta(t) = 0$$

The substitution of $\theta(t) = e^{mt}$ in this equation gives:-

$$e^{mt}\,(J\,m^2 + f\,m + K) = 0$$

Hence e^{mt} is a solution provided m is a root of the auxiliary equation:-

$$J\,m^2 + f\,m + K = 0$$

$$\therefore \quad m_1,\ m_2 = -\frac{f \pm \sqrt{f^2 - 4JK}}{2J}$$

Thus in the absence of load torque:-

$$\theta(t) = C_1\,e^{m_1 t} + C_2\,e^{m_2 t} + \theta_{ss}$$

When C_1 and C_2 are arbitrary constants.

When $f = 0$

$$m_1,\ m_2 = \pm j\sqrt{\frac{K}{J}}$$

Now the basic equation form for a second order system was given in Equation (6) as:-

$$\theta(t) = \theta_i \cos\sqrt{K'}\;t$$

Hence with the present system constants in the absence of friction:-

$$\theta(t) = \theta_i \cos\sqrt{\frac{K}{J}}\;t$$ which represents a free oscillation of natural frequency $\sqrt{\frac{K}{J}}$

Hence $\sqrt{\frac{K}{J}} = \omega_n$ where ω_n is the undamped natural frequency of the system.

$f = 0$

$$m_1,\ m_2 = \pm j\,\omega_n$$

$$\theta(t) = C_1 e^{+j\omega_n t} + C_2 e^{-j\omega_n t} + \theta_{ss}$$

which has been shown to be oscillatory.

$f^2 = 4JK$

$$m_1, m_2 = -\frac{f}{2J}$$

and $$\theta(t) = (C_3 + C_4 t)\, e^{mt} + \theta_{ss}$$

which represents an exponential response just free from oscillation, hence the damping is said to be critical.

Thus $f_c = 2\sqrt{JK}$ where f_c denotes critical damping.

We can now write:-

$$\zeta = \frac{\text{actual damping}}{f_c} = \frac{f}{2\sqrt{JK}}$$

and $$2\zeta\,\omega_n = \frac{f}{J}, \text{ since } \omega_n = \sqrt{\frac{K}{J}}$$

We may now rewrite Equation (9) in the absence of load torque:-

$$\theta(t) = \frac{(D^2 + 2\zeta\,\omega_n D)\,\theta_i(t)}{(D^2 + 2\zeta\,\omega_n D + \omega_n^2)}$$

The characteristic equation is now:-

$$(D^2 + 2\zeta\,\omega_n D + \omega_n^2)\,\theta(t) = 0$$

$$m_1, m_2 = -\zeta\,\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

Where $\omega_n\sqrt{1 - \zeta^2}$ is the natural frequency of damped oscillation of the system.

There are thus three forms of the roots of the characteristic equation depending upon the value of ζ:-

$\zeta < 1$ Roots conjugate complex with negative real parts. System underdamped.

$\zeta = 1$ Roots negative real equal. System critically damped.

$\zeta > 1$ Roots negative real and unequal. System overdamped.

Main interest is centred in the underdamped case when:-

$$m_1, m_2 = -\zeta\,\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

$$= a \pm jb$$

Hence $\theta(t) = e^{at}\left[C_3 e^{jbt} + C_4 e^{-jbt}\right] + \theta_{ss}$

But $e^{jbt} = \cos bt + j \sin bt$

Thus $\theta(t) = e^{at}\left[(C_3 + C_4)\cos bt + j(C_3 - C_4)\sin bt\right] + \theta_{ss}$

Since the solution must be real, we may write:-

$$\theta(t) = e^{-\zeta\omega_n t}\left[C_5 \cos \omega_n \sqrt{(1-\zeta^2)}\, t + C_6 \sin \omega_n \sqrt{(1-\zeta^2)}\, t\right] + \theta_{ss}$$

where $C_5 = (C_3 + C_4)$

$C_6 = j(C_3 - C_4)$

The constants must be determined on the basis of the initial conditions.

No purpose is served by considering the ensuing algebra here and the results for various disturbing functions are given in Table II.

A typical non-dimensional plot of output-input ratio against time $\omega_n t$ for various values of ζ is given in Figure 6.

When $\zeta\,\omega_n t = 4$, $e^{-\zeta\omega_n t} = 0.018$ and the system will have virtually reached the steady state.

4.5.3 Consideration of a numerical example

Assume $J = 49.10^{-4}$ slug ft^2

$f = 0.157$ ft lb/rad/sec

$K = 1.96$ ft lb/rad

The value of J would correspond to a motor of about 0.25 h.p.

If we assume a moderate input velocity of:-

$$\omega_i = 15 \text{ rad/sec}$$

Then $\omega_n = \sqrt{\dfrac{K}{J}} = \sqrt{\dfrac{1.96}{49.10^{-4}}} = 20$ rad/sec

and $\zeta = \dfrac{f}{2\sqrt{JK}} = \dfrac{0.157}{2\sqrt{49.10^{-4}.\,1.96}} = 0.8$

Time to reach steady state, $t = \dfrac{4}{\zeta\omega_n} = 0.25$ sec

and $\theta_{ss} = \dfrac{2\zeta\omega_i}{\omega_n} = 1.2$ rad

$$\theta_{ss} \simeq 69^\circ$$

This error is of course excessive, and is due to the system being insufficiently "stiff", as will be seen later when considering the harmonic response properties of this type of system.

Suppose we require $\theta_{ss} = 0.4$ rad

Then $$\frac{2\zeta\omega_i}{\omega_n} = 0.4 = \frac{f\omega_i}{K}$$

Hence $$K = 5.9 \text{ ft lb/rad}$$

and $$\zeta = 0.46$$

$$\omega_n = 34.8 \text{ rad/sec}$$

ζ is now rather low.

But $$\left.\begin{array}{l}\theta_{ss} \propto \dfrac{f}{K} \\ \zeta \propto \dfrac{f}{\sqrt{K}}\end{array}\right\} \text{For a fixed value of } J \text{ and } \omega_i$$

Hence if we double f and K, θ_{ss} remains unchanged, but ζ is increased by $\sqrt{2}$.

New values:-

$$J = 49.10^{-4} \quad \text{slug ft}^2$$

$$f = 0.314 \quad \text{ft lb/rad/sec}$$

$$K = 11.8 \quad \text{ft lb/rad}$$

$$\theta_{ss} = 0.4 \text{ as before}$$

$$\zeta = 0.65$$

$$\omega_n = 49.0 \quad \text{rad/sec}$$

The desired performance has been obtained by stiffening the system and increasing the friction coefficient, and it is instructive to consider the power dissipated in viscous friction.

$$\text{Power dissipated} = \frac{f\,\omega_i^2\,746}{550} \text{ watts}$$

$$= 96 \text{ watts}$$

Since the motor selected was only rated at 0.25 h.p. i.e. 186.5 watts, more than 50 per cent of the available power is wasted in viscous friction when this is set to satisfy the specification.

Thus a system employing viscous friction alone for damping is only of use for low power ratings and is inefficient; there is also the danger of large changes in damping occurring due to changes in ambient temperature.

We may summarise by stating that in the practical zero displacement error system when viscous friction and inertia are present, the system may have three forms of response depending upon the magnitude of the system parameters, namely underdamped when $\zeta < 1$, resulting in an oscillatory but stable response, critically damped when $\zeta = 1$ giving a response just free from oscillation, and overdamped when $\zeta > 1$ leading to a sluggish non-oscillatory response.

The achievement of a suitably damped but responsive system employing viscous friction alone as damping leads to an inefficient system.

4.6 The achievement of a damping term by means other than viscous friction damping

4.6.1 Output rate feedback

Returning to Equation (10), in the absence of load torque:-

$$\theta_o(t) = \frac{K\,\theta_i(t)}{[JD^2 + fD + K]}$$

We require a term $K_1 D\,\theta_o(t)$ which is not the result of viscous friction. One system is shown in Figure 7 employing output rate feedback by means of a tachometer generator driven from the output shaft, whose output voltage is proportional to output speed.

Thus:-

$$\frac{J\,d^2\theta_o(t)}{dt^2} + f\,\frac{d\,\theta_o(t)}{dt} = \text{Motor torque} = K\,\theta'(t)$$

$$\theta'(t) = \theta(t) - K_1\,\frac{d\,\theta_o(t)}{dt}$$

$$\frac{J\,d^2\theta_o(t)}{dt^2} + (f + K\,K_1)\,\frac{d\,\theta_o(t)}{dt} = K\,\theta(t)$$

$$\therefore \quad \theta_o(t) = \frac{K\,\theta_i(t)}{[J\,D^2 + (f + K\,K_1)\,D + K]}$$

The effective friction coefficient has thus been increased from f to $(f + K\,K_1)$, and it may be noted that the additional factor is a function of quantities capable of reasonable regulation. Other more subtle means are possible for obtaining the same effect such as extracting the back e.m.f. of the drive motor, or using the feedback properties of a falling torque-speed characteristic motor. This system has several disadvantages, tending to increase the error for velocity input and lowering the rate of response.

4.6.2 Modification to the controller characteristic - proportional plus error derivative control, or "error rate damping", and addition of an integral term

The preceding discussion has emphasised the importance of the error - output chain which is a primary factor in the system response behaviour and introduced the conception of a controller whose performance may be modified to provide the desired system characteristic; in the foregoing the controller may be taken to include the amplifier-motor system between error and output.

The controller forms a convenient unit in which to adjust the system, particularly if it includes an electronic amplifier. In the latter case correcting networks may be embodied which can be readily adjusted, even under operating conditions if required.

Our previous controller characteristic was:-

$$\text{Torque} = K\,\theta(t)$$

We now consider a generalised characteristic of the form:-

$$\text{Torque} = \left[K\,\theta(t) \pm \ell \frac{d\,\theta(t)}{dt} \pm m \frac{d^2\theta(t)}{dt^2} \right]$$

The governing equation is then:-

$$\theta(t) = \frac{(J\,D^2 + fD)\,\theta_i(t)}{(J\,D^2 \pm m\,D^2 + fD \pm \ell D + K)}$$

$$= \frac{(J\,D^2 + fD)\,\theta_i(t)}{\left[(J \pm m)\,D^2 + (f \pm \ell)\,D + K\right]}$$

Inspection of preceding governing equations shows that f must be kept to a minimum to reduce power wastage and the ideal arrangement would be to have $f = 0$ and ℓ set positive at the appropriate value. Reduction in inertia leads to improved response and although resulting also in a reduction in ζ can be compensated for by adjustment to ℓ. Thus m must be set negative, but not such that the effective inertia is reduced to a very low value, when instability would result.

If f is reduced to zero the steady state error for a step input of velocity ω_i is also zero and the transient solution is:-

$$\theta \frac{\omega_n}{\omega i} = \frac{e^{-\zeta\omega_n t} \sin\left[\omega_n\sqrt{1 - \zeta^2}.t\right]}{\sqrt{1 - \zeta^2}}$$

which compares with the solution for viscous friction damping alone:-

$$\theta(t)\frac{\omega_n}{\omega_i} = 2\zeta - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin\left[\omega_n\sqrt{1 - \zeta^2}.t + \phi\right]$$

where $2\zeta = \theta_{ss}\frac{\omega_n}{\omega_i}$ $\zeta < 1$

$$\phi = \tan^{-1}\frac{2\zeta\sqrt{1 - \zeta^2}}{2\zeta^2 - 1}$$

These are plotted for comparison in Figure 8. Thus the addition of derivative of error known also as "error rate damping" provides a ready method of adjusting the response of the system, although the second derivative of error is difficult to obtain and seldom used. This method is free from the disadvantages mentioned in relation to output rate feedback.

Hence the torque equation is (in the absence of viscous friction):-

$$\text{Torque} = K\,\theta(t) + \ell\frac{d\theta(t)}{dt}$$

i.e. proportional + derivative.

The action may be visualised by consideration of the action of the two torque components after application of a step input of velocity ω_i to a system with $\zeta < 1$:-

(a) Immediately after application of the input step the error rate is equal to input velocity and the rate term provides a maximum assisting torque.

(b) As the output speeds up the excess of input speed over output speed diminishes, thereby reducing the error rate torque.

(c) As the output accelerates further it overtakes the input and the error rate and its torque become negative, thus reducing the overshoot.

The system is also known as "anticipation control", but this is really a misnomer, the action being an augmenting or boosting action. True anticipation requires, for example, a feed forward to the controller from the input in such a way that the controller can take action before the generation of an error or derivative of error.

The system is additionally known as a "phase advance" system, for reasons which will be considered later.

When derivative and viscous friction damping are both employed the response characteristics are between the basic responses of the independent systems.

Presence of viscous friction re-introduces the problem of eliminating steady state errors and by considering preceding sections it may be deduced that these may be eliminated by the addition of an integral term, thus giving the overall controller characteristic:-

$$\text{Torque} = \underbrace{K\,\theta(t)}_{\text{proportional}} + \underbrace{\ell\,\frac{d\,\theta(t)}{dt}}_{\text{derivative}} + \underbrace{K'\int\theta(t)\,dt}_{\text{integral}}$$

As has been considered previously integral control tends to a destabilising effect and hence the various parameters must be carefully selected for the best solution.

4.7 The stability of simple systems of higher order

As the control becomes more complex it is not easy to deduce the stability of the system; the systems discussed above have been limited to second order differential equations which can result in a neutrally stable but not a divergent oscillatory response. With higher order systems the latter is a possibility and since solution of the governing differential equation becomes a major undertaking there is clearly need for a short cut to determination of the system stability which does not require a complete solution of the differential equation.

It may be deduced from previous sections that a system may be described by a general equation of the form:-

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots\ldots\ldots + a_o)\ \theta_o(t) = f(t)$$

and that the stability of the system is dependent upon the transient solution, i.e. the complementary function. The stability depends then on the roots of the auxiliary equation which may be written:-

$$(a_n m^n + a_{n-1} m^{n-1} + \ldots\ldots\ldots + a_o) = 0$$

$$\text{or } (m_1 - \gamma_1)(m_2 - \gamma_2) \ldots\ldots (m_{n-1} - \gamma_{n-1})(m_n - \gamma_n) = 0$$

where $\gamma_1, \gamma_2 \ldots\ldots \gamma_k \ldots\ldots \gamma_n$ are the roots of the auxiliary equation.

If γ_k is real the corresponding solution is exponential:-

$$\theta_o(t) = A\, e^{\gamma_k t}$$

and is damped provided γ_k is negative.

If γ_k is complex the roots will exist in pairs of the form $(\alpha_k \pm j\,\beta_k)$ giving solutions of the form:-

$$\theta_o(t) = A' e^{\alpha_k t} \cos(\beta_k t + \psi_k)$$

which again is damped provided α_k is negative.

We can now write down conditions for stability based on the coefficients of the differential or auxiliary equation:-

(1) The real parts of the roots of the auxiliary equation must be negative.

This may be interpreted, if we exclude neutrally stable systems such as the basic second order control, as:-

(1a) All powers of D or m must be present.

(1b) The coefficients of D or m must have the same sign.

(2) The coefficients of D or m must fulfil certain algebraic relationships.

The criterion has been given by Routh for n = 3:-

$$a_1\, a_2 > a_0\, a_3$$

and when n = 4 there must be satisfaction additionally of the relation:-

$$a_1\, a_2\, a_3 > a_0\, a_3^2 + a_4\, a_1^2$$

It is important at this juncture to appreciate that whilst this criterion establishes whether a system is stable or not it gives no indication of the degree of stability.

The significance of the above is indicated by consideration of the location of the roots in the m plane, when it will be seen that the roots must lie in the left-hand half of the plane for stability, that no conjugate roots may lie on the imaginary axis and that no multiple order zero roots may lie at the origin (Figure 9).

Figure 9 also serves to introduce the conception that there are only six basic forms which the roots of the characteristic or auxiliary equation may take, of which only two are basically favourable from a stability viewpoint.

The root forms are:-

$$\left.\begin{array}{lll} \text{Root 1} & m = & +\alpha \\ \text{Root pair 2} & m = & +\alpha \pm j\beta \\ \text{Root 3} & m = & 0 \\ \text{Root pair 4} & m = & \pm j\beta \end{array}\right\} \text{Unfavourable}$$

$$\left.\begin{array}{lll} \text{Root 5} & m = & -\alpha \\ \text{Root pair 6} & m = & -\alpha \pm j\beta \end{array}\right\} \text{Favourable}$$

It should be emphasised that there are systems with multiple loops which may have roots in the right-hand half of the m plane and yet are stable, provided the main loop remains closed. The standard works such as Reference 1 should be consulted for further consideration of these.

4.8 Response of a typical system to a harmonic oscillation of input

The step input function considered so far has represented the extreme and generally unrealisable case of an instantaneous change in command and although the resulting response of output, error, etc. has been considered on a time basis, no consideration has been given to the response of the system on a frequency and phase basis as the result of a periodic function of input.

It will have been inferred from results given above that since some of the systems considered have exhibited oscillatory tendencies, they will when subjected to an appropriate harmonic input exhibit resonance properties leading to various amplitude and phase shift errors; the latter may of course result in excessive or dangerous errors in control operation.

4.8.1 Periodic and harmonic motion

When a function is repeated at a regular time interval it is said to be periodic, the simplest form of periodic motion being a sine or cosine function, which is a harmonic periodic motion and has the form:-

$$\theta = \hat{\theta} \sin \omega t \text{ or } \theta = \hat{\theta} \cos \omega t$$

where $\hat{\theta}$ is the maximum value of amplitude, and the periodic time of motion

$$T = \frac{2\pi}{\omega}$$

where T is the reciprocal of the frequency of oscillation in cycles per unit time.

Now the significance of the harmonic motion is revealed by the Fourier theorem which states that any periodic motion of circular frequency ω may be represented by the vector summation of a series of harmonic functions of frequency ω, 2ω, 3ω, etc. and appropriate phase relation, thus justifying the use of a sinusoidal function as the base form of periodic motion.

Then:

$$f(t) = A_0 + A_1 \sin(\omega t + \psi_1) + A_2 \sin(2\omega t + \psi_2) + \ldots\ldots\ldots$$

where ψ_1, ψ_2, etc. are the appropriate phase angles.

The Fourier series may also be written:-

$$f(t) = a_1 \sin \omega t + a_2 \sin 2\omega t + \dots a_n \sin n\omega t + b_0 + b_1 \cos \omega t + \dots + b_n \cos n\omega t + \dots$$

where $A_n^2 = a_n^2 + b_n^2$

$$\tan \psi_n = \frac{b_n}{a_n}$$

Many highly developed techniques exist for determination of the various coefficients, i.e. for performing harmonic analysis. Mechanical harmonic analysers are now common tools in industry.

4.8.2 System linearity and the principle of linear superposition

Much servo-system theory is based upon the assumption of linearity in the system, which in effect means that the input-output relationship is continuous and independent of the magnitude of input. Whilst this is rarely realised in practice it serves as a working hypothesis which is then capable of mathematical analysis. Thus the response of a linear system to a harmonic input is a harmonic output of identical frequency, but differing possibly in amplitude and phase; the response of a linear system to a complex input is thus the vector sum of the individual responses of the system to the harmonic components obtained by Fourier analysis of the input.

This is then an example of the principle of linear superposition.

The Fourier theorem thus provides the link between harmonic and step function response.

4.8.3 Zero displacement error system with harmonic input

For the zero displacement error system considered above the governing equation has been shown to be:-

$$(D^2 + 2\zeta\,\omega_n D + \omega_n^2)\,\theta_o(t) = \omega_n^2\,\theta_i(t)$$

$$\theta_o(t) = \frac{\omega_n^2\,\theta_i(t)}{(D^2 + 2\zeta\,\omega_n D + \omega_n^2)}$$

If $\theta_i(t) = B \cos \omega t$

$$\theta_o(t) = \frac{\omega_n^2\,B \cos \omega t}{(D^2 + 2\zeta\,\omega_n D + \omega_n^2)}$$

Now the solution to this equation will take the form of a transient solution added to a forced solution as previously. In a stable system the transient solution will die out leaving the forced solution, i.e. the particular integral only, which by normal operational methods gives:-

$$\theta_o(t) = \frac{B}{\sqrt{(1-d^2)^2 + (2\zeta d)^2}} \cos\left(\omega t - \tan^{-1}\frac{2\zeta d}{1-d^2}\right)$$

where $$d = \frac{\omega}{\omega_n}$$

This solution is known as the steady state response to a harmonic input and must not be confused with the alternative use of the term steady state to depict a condition of rest or equilibrium.

The above solution represents an output vector with the properties:-

$$\left.\begin{aligned} \text{Amplitude} &= \frac{B}{\sqrt{(1-d^2)^2 + (2\zeta d)^2}} = B.M_M \\ \text{Phase} &= -\tan^{-1}\frac{2\zeta d}{(1-d^2)} = \psi \end{aligned}\right\} \text{Relative to the input vector}$$

where
$$\theta_i(t) = B\cos\omega t$$
$$\theta_o(t) = B\,M_M\cos(\omega t + \psi)$$

It will be seen that in the region $d = 1$, for small values of ζ, $M_M = \frac{1}{\sqrt{(1-d^2)^2 + (2\zeta d)^2}}$ is considerably greater than unity, hence M_M is known as the dynamic magnifier. As $\zeta \to 0$ $M_M \to \infty$ at $d = 1$, and $\psi = -\frac{\pi}{2}$ radians; as $d \to \infty$ $\psi \to -\pi$ radians.

These results are plotted in Figure 10 for various values of ζ and it will be noted that the maximum value of M_M occurs at a value of d, $d < 1$.

4.8.4 Zero displacement error system with error rate damping and harmonic input

The response of this system to unit harmonic response input $1.\cos\omega t$ is given by the particular integral of:

$$\theta_o(t) = \frac{\omega_n^2\cos\omega t - 2\zeta\omega\,\omega_n\sin\omega t}{(D^2 + 2\zeta\,\omega_n D + \omega_n^2)}$$

Hence $$\theta_o(t) = \sqrt{\frac{1+(2\zeta d)^2}{(1-d^2)^2 + (2\zeta d)^2}}\,.\,\sin\left(\omega t + \tan^{-1}2\zeta d - \tan^{-1}\frac{2\zeta d}{1-d^2}\right)$$

where $$M_M = \sqrt{\frac{1+(2\zeta d)^2}{(1-d^2)^2 + (2\zeta d)^2}}$$

$$\psi = -\tan^{-1}\frac{2\zeta d}{1-d^2} + \tan^{-1}2\zeta d$$

Comparing this solution with that for the system without error rate damping in 4.8.3, it will be noted that "the phase has been advanced" by $\tan^{-1}2\zeta d$, and that as $d \to \infty$ $\psi \to -\frac{\pi}{2}$.

These results are plotted in Figure 11.

It will be noted that $M_M = 1$ for all values of ζ at $d = \sqrt{2}$.

5.0 Correlation between transient response and harmonic response

It will be noted from comparison between Figure 6, and Figure 10 and Figure 11, that there is a close relation between the properties of the transient response and the harmonic response, and considerable significance attaches to this in the design and improvement of control systems. For example peaks in the harmonic response of a complex system would be a guide to the relative damping of the natural modes of oscillation of the transient response.

As a general value $M_M = 1.3 - 1.4$ is often considered to be a reasonable value for a practical system.

5.1 The use of complex numbers to solve for functions of time and frequency

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

If we assume

$$\theta_i(t) = 1.\sin \omega t = \text{Imaginary} \left\{ e^{j\omega t} \right\}$$

$$\theta_o(t) = A.\sin(\omega t + \psi) = \text{Imaginary} \left\{ Ae^{j(\omega t + \psi)} \right\}$$

If we return to the governing equation:-

$$(JD^2 + fD + K)\,\theta_o(t) = K\theta_i(t)$$

By rewriting this equation in complex form:-

$$(-J\omega^2 + j\omega f + K)\,Ae^{j(\omega t + \psi)} = Ke^{j\omega t}$$

$$\therefore \quad Ae^{j\psi} = \frac{\omega_n^2}{(-\omega^2 + 2j\omega\zeta\omega_n + \omega_n^2)}$$

Thus we may write

$$Ae^{j\psi} = \frac{\theta_o}{\theta_i}(j\omega) = \frac{\omega_n^2}{\left[(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2\right]}$$

where $\frac{\theta_o}{\theta_i}(j\omega)$ is a vector ratio of the output to input expressed as a function of the complex variable $j\omega$.

It is usually more convenient to work in non-dimensional quantities, hence writing $d = \frac{\omega}{\omega_n}$:-

$$\frac{\theta_o}{\theta_i}(jd) = \frac{1}{(1 - d^2) + 2\zeta jd}$$

$$\equiv \frac{(1 - d^2)}{(1 - d^2)^2 + (2\zeta d)^2} - j\,\frac{2\zeta d}{(1 - d^2)^2 + (2\zeta d)^2}$$

$$\equiv \quad C \quad + \quad j\,D$$

This represents a vector of magnitude $\sqrt{C^2 + D^2}$ having a phase angle $\tan^{-1}\frac{D}{C}$ (Figure 12).

Thus:-

Vector magnitude $\dfrac{1}{\sqrt{(1 - d^2)^2 + (2\zeta d)^2}}$

Vector phase $-\tan^{-1}\dfrac{2\zeta d}{1 - d^2}$

The locus of the vector $\frac{\theta_o}{\theta_i}(jd)$ is plotted in Figure 12 for various values of ζ.

The solution as a function of time is:

$$\frac{\theta_o}{\theta_i}(t) = \frac{1}{\sqrt{(1 - d^2)^2 + (2\zeta d)^2}} \sin\left(\omega t - \tan^{-1}\frac{2\zeta d}{1 - d^2}\right)$$

which was obtained in 4.8.3.

By the use of complex numbers we have been able to obtain a vector ratio of output to input as a function of the complex frequency variable $(j\omega)$ or (jd) which describes the behaviour of the system when it is subjected to a harmonic input in the absence of transient terms; i.e. the steady state response to a harmonic input. Under these conditions if we write:-

$$\theta_o(t) = Q\,(D)\,\theta_i(t)$$

it is valid to substitude $j\omega$ for D and obtain the vector ratio of the complex variable:-

$$\frac{\theta_o}{\theta_i}(j\omega) = Q\,(j\omega)$$

It is a simple matter to obtain from this relationship the solution to the original governing equation as a time function in response to the harmonic input.

6.0 The conception of a transfer function

Introduction of the concept of a transfer function has been deliberately avoided in the earlier part of the discussion to prevent confusion resulting from the number of ways in which it may be defined; at the commencement of the consideration of control systems the system appeared as a number of elements which combined to establish a differential equation from which the response was calculated. Later on it was shown that control systems could be conveniently grouped into orders and the conception of a controller was introduced. There was thus established the idea of an input to an element which was operated on within the element to provide an output and the general relation obtained:-

$$\theta_o(t) = Q(D)\,\theta_i(t)$$

where $Q(D)$ represents an operator which modifies the input to provide the output. This quantity is called the transfer function of the element, (not, it may be noted, the transfer function of the system). Alternatively we may transform to the complex form:-

$$\frac{\theta_o}{\theta_i}(j\omega) = Q(j\omega)$$

where $Q(j\omega)$ represents the element transfer function as a vector based on the complex variable $j\omega$. Reference to Equation (7) will recall that the field system of the generator and amplifier represented a simple lag which had the transfer function $\left(\frac{K_1}{1+\tau D}\right)$ and that in general the transfer function has the form $KG(D)$ or $KG(j\omega)$ where K represents the gain, or part of the transfer function invariant with frequency, and $G(j\omega)$ represents the complex part varying with frequency.

We now have a means of representing the behaviour of an element in a concise and convenient manner, so that its input-output performance is always known; this is particularly convenient in a complex system such as Figure 13, where

$$\theta_2 = Q_1.\theta_1$$

$$\theta_3 = Q_2.\theta_2$$

$$\theta_4 = Q_3.\theta_3$$

The important conclusion which follows is that by multiplying transfer functions we can obtain the overall input-output relation:-

$$\theta_4 = Q_1\,Q_2\,Q_3\,.\,\theta_1$$

or $$\frac{\theta_4}{\theta_1}(j\omega) = K_1G_1(j\omega).K_2G_2(j\omega).K_3G_3(j\omega)$$

$$= KG(j\omega)$$

where $$K = K_1.K_2.K_3$$

$$G(j\omega) = G_1(j\omega).G_2(j\omega).G_3(j\omega)$$

The foregoing concerns elements or groups of open chain elements, but our main concern is with closed loop systems; we may depict this in block diagram form (Figure 13) where $Q = Q_1.Q_2.Q_3$ etc. depicts the element transfer functions grouped together. Now if a signal, say ε, originates in the loop it will be transformed on arrival at the same point after traversing the system as:-

$$-\left\{Q_1.Q_2.Q_3\text{ etc.}\right\}\varepsilon = -Q\,\varepsilon$$

where Q represents the loop transfer function and is descriptive of the functioning of the complete system.

So that:-

$$\theta_o(t) = Q(D)\ \theta(t)$$

and since

$$\theta(t) = \theta_i(t) - \theta_o(t)$$

$$\theta_o(t) = \frac{Q(D)}{1 + Q\ (D)}\ \theta_i(t)$$

We have now effected a subtle transfer of emphasis from the input-output behaviour of the system to the system response on the basis of a loop transfer function which relates output to error (not to input).

Thus:-

$$\frac{\theta_o}{\theta}\ (j\omega) = KG(j\omega)$$

$$\theta(j\omega) = \theta_i(j\omega) - \theta_o(j\omega)$$

$$\frac{\theta_o}{\theta_i}\ (j\omega) = \frac{KG(j\omega)}{1 + KG(j\omega)}$$

where $KG(j\omega)$ is the loop transfer function; in American literature this is known simply as the transfer function of the closed loop system, or merely the transfer function.

Since

$$\frac{\theta}{\theta_i}\ (j\omega) = \frac{1}{1 + KG(j\omega)}$$

we obtain the important characteristic equation for the closed loop system with unity feedback:-

$$1 + KG(j\omega) = 0$$

It may be noted in passing that if the closed loop system is opened at a convenient point and a signal sent round the loop, measurement of the signal on return to the same point gives a measurement of the loop transfer function. This may be done by feeding in a range of harmonic signals at different frequencies and measuring the dynamic magnifier and phase angle of the returning signal relative to the input, thus enabling the harmonic response locus to be plotted, i.e. the locus of $\frac{\theta_o}{\theta}\ (j\omega)$; the latter function is called the harmonic response function.

7.0 Use of the Laplace transformation

The Laplace transform method is an operational calculus approach which enables impulsive and discontinuous functions to be dealt with, provides the transient and steady state solutions simultaneously and is readily able to take account of initial conditions. The method is superior to the classical operator technique for the solution of high order equations, and by use of the transform, conversion of the differential equation in the time domain to a function of a complex variable can be effected in which this function can be manipulated by the normal laws of algebra; this functional relationship can be returned to the time domain by inverse transformation.

The method is used in American literature almost exclusively, but although it is a powerful method of considerable value, its use in elementary problems may be no advantage over the classical methods already discussed. Due to substitutions which can be effected when certain restrictions are applied to the system, casual inspection of American literature may lead to considerable confusion in the mind of the reader as to the use of the method and even in the definition of properties of the system such as transfer functions. Indiscriminate substitution of the Laplace parameter may also lead to error unless initial conditions are reconciled and a true appreciation of the method obtained. It is for this reason that consideration of the method has been left to this stage in the paper.

A function of time and of the complex variable are related by definition as:-

$$f(p) = \mathcal{L}\, f(t)$$

Where $f(p)$ is a function of the Laplace parameter p*

p is a complex variable of the form $\alpha + j\omega$

$\mathcal{L}$ denotes application of the transform integral

$f(t)$ is a function of time.

The transform integral is:-

$$f(p) = \int_0^\infty e^{-pt} f(t)\, dt = \mathcal{L}\, f(t)$$

There are certain limitations to unrestricted application of the above which do not concern us here.

If we consider a unit step function of time:-

$$f(t) = 1$$

$$f(p) = \int_0^\infty e^{-pt} . 1\, dt$$

$$= -\frac{1}{p}\Big[e^{-pt}\Big]_0^\infty = \frac{1}{p}$$

Hence $\mathcal{L}\big[f(t)\big] = f(p) = \dfrac{1}{p}$ for unit step function.

For an exponential decay:-

$$f(t) = e^{-\omega t}$$

---

* The Laplace parameter p is denoted by s in American literature

$$\mathcal{L}(e^{-\omega t}) = \int_0^\infty e^{-(p+\omega)t}\,dt = \frac{1}{p+\omega}$$

For a sinusoidal function:-

$$\mathcal{L}(\sin \omega t) = \int_0^\infty \sin \omega t\, e^{-pt}\,dt$$

$$= \frac{1}{2j}\int_0^\infty (e^{j\omega t} - e^{-j\omega t})\, e^{-pt}\,dt$$

$$\mathcal{L}(\sin \omega t) = \frac{1}{2j}\left(\frac{1}{p - j\omega} - \frac{1}{p + j\omega}\right) = \frac{\omega}{p^2 + \omega^2}$$

Some frequently used transform pairs are given in Table III.

It follows that:-

$$\mathcal{L}\left[K\, f(t)\right] = K \quad f(t) = K\, f(p)$$

$$\mathcal{L}\left[f_1(t) + f_2(t)\right] = f_1(p) + f_2(p)$$

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = p\, f(p) - f(0\,+)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = p^2\, f(p) - p\, f(0\,+) - \frac{df}{dt}(0\,+)$$

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{f(p)}{p} + \frac{1}{p}\left[\int f(t)\,dt\right]_{t\,=\,(0\,+)}$$

The terms $f(0\,+)$ may occasion some surprise but is necessary due to the use of discontinuous functions and means the value of $f(t)$ as t approaches zero from the positive direction. Consider for example a unit step function occurring at $t = 0$ then at $t = (0\,-), f(0\,-)$ is zero, but at $t = (0\,+)$, $f(0\,+)$ is unity.

Various theorems and applications of the initial and final value theorems enable the relationships to be manipulated and short cuts effected (Reference 2).

Inverse transformation denoted by $^{-1}$ enables functions of p to be converted back to the time domain e.g.:-

$$\mathcal{L}^{-1}\left(\frac{\omega}{p^2 + \omega^2}\right) = \sin \omega t$$

$$\mathcal{L}^{-1}\left(\frac{1}{p + \omega}\right) = e^{-\omega t}$$

We return now to consideration of the simple system denoted by Equation (11) in the absence of load torque:-

$$D\,(JD + f)\,\theta_o(t) = K\,\theta(t)$$

Since

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$2\zeta\,\omega_n = \frac{f}{J}$$

$$D\,(D + 2\zeta\,\omega_n)\,\theta_o(t) = \omega_n^2\,\theta(t)$$

$$D^2\,\theta_o(t) + 2\zeta\,\omega_n\,D\,\theta_o(t) = \omega_n^2\,\theta(t)$$

We can now take the Laplace transform of each term to obtain:-

$$\mathcal{L}\left[\frac{d^2\theta_o}{dt^2}\right] = p^2\,\theta_o(p) - p\,\theta_o\,(0+) - \frac{d\theta_o}{dt}\,(0+)$$

$$\mathcal{L}\left[2\zeta\,\omega_n\,\frac{d\theta_o}{dt}\right] = 2\zeta\,\omega_n\,p\,\theta_o(p) - \theta_o\,(0+)$$

$$\mathcal{L}\left[\omega_n^2\,\theta(t)\right] = \omega_n^2\,\theta(p)$$

Thus

$$\theta_o(p) = \underbrace{\frac{\omega_n^2}{p\left[p + 2\zeta\,\omega_n\right]}\,\theta(p)}_{\text{transfer function}} + \underbrace{\frac{\left[p\,\theta_o\,(0+) + \frac{d\theta_o}{dt}\,(0+) + 2\zeta\,\omega_n\,\theta_o\,(0+)\right]}{p\left[p + 2\zeta\,\omega_n\right]}}_{\text{initial condition transform}}$$

If we consider $\theta_o(p)$ in the form:-

$$\theta_o(p) = \frac{A(p)}{B(p)}$$ we may eventually resolve the expression by partial fractions into the form:-

$$\frac{A(p)}{B(p)} = \frac{K_1}{(p - \gamma_1)} + \frac{K_2}{(p - \gamma_2)} + \ldots.$$

provided $\frac{A(p)}{B(p)}$ is proper and contains no repeated roots, or if $\frac{A(p)}{B(p)}$ is improper or contains repeated roots then we proceed by the normal rules of partial fractions. The inverse transform may then be obtained for each fraction individually, considerable use being made of the property:-

$$\mathcal{L}^{-1} \frac{K_k}{(p - \gamma_k)} = K_k e^{\gamma_k t}$$

The constants are found from the parameters of the system and the final solution is obtained as a function of time by substitution of the initial conditions. Use of the Laplace transform method for simple systems requires at least as much work as the classical method for the solution of a differential equation, but is a powerful aid in solving high order systems.

If we assume that the initial conditions are zero we obtain:-

$$\theta_o(p) = \frac{\omega_n^2 \ \theta(p)}{p\left[p + 2\zeta \ \omega_n\right]}$$

which may be compared with Equation (11) when we obtained:-

$$\theta_o(t) = \frac{\omega_n^2 \ \theta(t)}{D\left[D + 2\zeta \ \omega_n\right]}$$

We may write:-

$$\theta_o(p) = \frac{\dfrac{\omega_n^2}{2\zeta \ \omega_n}}{p\left[\dfrac{p}{2\zeta \ \omega_n} + 1\right]} \theta(p)$$

Hence if we write $$K_v = \frac{\omega_n}{2\zeta} = \text{"velocity constant"}$$

and $$\tau = \frac{1}{2\zeta \ \omega_n} = \text{a "time constant"}$$

$$\theta_o(p) = \frac{K_v \ \theta(p)}{p \ (1 + p\tau)}$$

It is clear then that when the initial conditions are zero we may write for a harmonic input:-

$$\frac{d}{dt} \equiv D \equiv p \equiv j\omega$$

and that our transfer function may be considered as:-

$\theta_o = Q(D) \ \theta$ in terms of the differential

operator, or:-

$\frac{\theta_o}{\theta}(p) = Q(p) = KG(p)$ in terms of the Laplace

parameter, or:-

$\frac{\theta_o}{\theta}(j\omega) = KG(j\omega)$ in complex vector ratio form.

Since the Laplace transform method is favoured in the U.S.A. the transfer function is defined there as:-

$$KG(p) = \frac{\text{Laplace transform of the output}}{\text{Laplace transform of the error}}$$

which if encountered without the background discussed above, puts an apparently complicated construction on a basically simple function.

8.0 Analysis and synthesis

Opportunity will now be taken to make an appreciation of the material covered so far in relation to control system performance.

In the first part of the discussion the system existed as an assembly of rather unrelated elements and solution of the governing differential equation yielded the transient response as a time function relation between output and input. The conception of order of control was introduced and it was established that the majority of systems could not have the perfect response but were a compromise. The presence of friction and inertia in a practical system was shown to raise the order of the differential equation and to introduce oscillatory behaviour depending upon the value of damping present. Consideration of a typical system employing viscous friction damping alone was shown to be exceedingly inefficient, and alternative methods of introducing damping were considered such as output rate feedback, and a derivative of error term or error rate damping; the addition of a negative second derivative term to reduce the effective inertia was also considered together with an integral term for removal of steady state error. This introduced the conception of a controller between error input and output, comprising for example an amplifier and motor to provide the required circuits and power amplification for obtaining the desired response of output to error. Consideration was given to determining the stability of simple systems without the necessity for a complete solution, by inspection of the coefficients of the governing equation and the important point was made that Routh's criterion did not determine the degree of stability. System response to a harmonic input was examined and the correlation existing between the form of response to this and to step functions of input noted. The block interpretation of systems led to the idea of an element transfer function defining completely the input-output function of the element and introduced the important loop transfer function of a closed loop system relating output to error, which contained all the information required about the system. Finally the vector locus and Laplace transform were introduced and the identity of $\frac{d}{dt}$, D, $j\omega$, and p established for a system subject to a harmonic input in which the initial conditions were zero; the characteristic equation of the closed loop system with unity feedback was shown to be $1 + KG(p) = 0$. It was also shown that the loop transfer function could be established by opening the loop at any point, injecting a signal and noting the modifications to it introduced in its passage back to the starting point; if the signal comprised a harmonic input whose frequency could be varied, it was possible to determine the loop response by measurement of the dynamic magnification and phase of the returning signal; from this the harmonic response locus could be plotted.

The crux of the control system problem has been shown to lie then not in an analysis of the performance of a typical system in the form of the input-output response, but in the combined problem of establishing the design of a system which will have the required response and then ensuring that its response will in fact be the one desired. This is the problem of

synthesis and much of the confusion which a newcomer to the subject experiences is his inclination to work on the basis of analysis, when the wealth of control system technique is devoted to synthesis; this results in his being confronted with apparently abstract quantities such as the vector locus of the complex variable $j\omega, \frac{\theta_o}{\theta}(j\omega)$, rather than input-output functions such as $\theta_o(t)$ and $\theta_i(t)$. It may appear as if solution of the differential equations or algebraic treatment of the roots of $1 + KG(p) = 0$ must enable synthesis to be effected, but in the case of a complex system this involves vast amounts of computing, and although the effect of changes in the system can be calculated, it is virtually impossible to obtain the optimum arrangement of the complete control system except by chance. Considerable effort has been expended on analysis of the transient response, which for many years was the only tool available and a number of techniques and graphical methods developed for easing the determination of system response and stability.

Modern methods of synthesis are founded primarily on a frequency response basis, using the correlation between transient and harmonic response to ensure that the transient response will be satisfactory. Much of the technique is based upon properties of the loop transfer function and its vector locus $KG(j\omega)$; an alternative version of the latter known as the inverse transfer function locus $\frac{1}{KG(j\omega)}$ or $KG^{-1}(j\omega)$ is sometimes more convenient. A major part of the work comprises adjusting the K function and modifying the $G(j\omega)$ locus until the desired performance is achieved. The degree of stability is also established by these techniques.

One major advantage of this approach is that by dealing in vector loci, system elements may be dealt with on a transfer function basis by determining the element transfer function experimentally, even though the differential equation of it cannot be formulated mathematically. This method lends itself readily to the use of analogues; if for example the complex machine or engine is to have an electrical controller, the transfer function may be determined experimentally and an electrical analogue designed to give the same response as the actual engine. Calculations and synthesis can proceed on the basis of the measured response and checks made on the actual control apparatus employing the analogue rather than the actual machine.

9.0 The design process

The first requirement in the design is a specification defining the response under certain basic conditions which include:-

(1) The degree of stability required following a step or pulse disturbance.

(2) The error $\theta(t)$ which is acceptable for specified variations of input $\theta_i(t)$ or load $\Gamma_L(t)$.

(3) The phase shift between input and output which is allowable and the frequency band width to be covered.

With a knowledge of the particular function of the control system the designer may then be able to lay down the basic properties, e.g. order of control etc.

The procedure then comprises:-

(4) Construction of a block diagram representing the physical elements.

(5) Formulation of the element transfer functions by calculation or experiment; where non-linearities exist the system must be linearised by the accepted mathematical methods.

(6) The system may then be analysed and the response determined, thus completing the analysis.

(7) Should there be undesirable features present in the response, the techniques of synthesis must be applied and the system adjusted to give the required response.

It is not the purpose of this Report to consider the design of control systems except in so far as this helps to clarify the engine response rate problem, but some consideration must be given to the basic properties of the transfer function locus on the complex plane, i.e. $KG(j\omega)$ if the picture is to be complete.

10.0 Some basic properties of the $KG(j\omega)$ locus and its inverse locus $KG^{-1}(j\omega)$

It has been established that for the closed loop system with unity feedback defined by:-

$$\theta_i(j\omega) - \theta_o(j\omega) = \theta(j\omega)$$

$$\frac{\theta_o}{\theta}(j\omega) = KG(j\omega)$$

$$\frac{\theta_i}{\theta}(j\omega) = 1 + KG(j\omega)$$

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{KG(j\omega)}{1 + KG(j\omega)}$$

$$\frac{\theta}{\theta_o}(j\omega) = KG^{-1}(j\omega)$$

$$\frac{\theta_i}{\theta_o}(j\omega) = \frac{1 + KG(j\omega)}{KG(j\omega)}$$

$$= 1 + KG^{-1}(j\omega)$$

10.1 Graphical determination of $\frac{\theta_o}{\theta_i}(j\omega)$ from the $KG(j\omega)$ locus

A representative locus of $KG(j\omega)$ is given in Figure 14, the vector being of amplitude OB and phase $-\phi$. Point A, which will be shown to be very important, is at -1 on the real axis, i.e. the $(-1 + 0j)$ point.

Then AB represents the magnitude of the vector $1 + KG(j\omega)$, and phase $-\varepsilon$.

$$\text{Thus} \quad \frac{OB}{AB} = \frac{KG(j\omega)}{1 + KG(j\omega)} = \frac{\theta_o}{\theta_i}(j\omega)$$

and the phase angle = angle OBA = $\phi - \varepsilon$.

The locus of $\frac{\theta_o}{\theta_i}(j\omega)$ may then be re-plotted (Figure 14) and it will be recalled that the magnitude of this vector is the dynamic magnifier M_M. It may be noted that the triangle ABO represents vector AO $\equiv \theta$, vector OB $\equiv \theta_o$ vector AB $\equiv \theta_i$ to an appropriate scale.

10.2 Stability and the (- 1 + 0j) point, the Nyquist diagram

When the locus passes through the (-1 + 0j) point it follows that a signal originating in the loop will pass round the system and appear at the starting point as a signal of equal amplitude and in phase with it. The signal is then self-sustaining without the need for an input. Thus the (-1 + 0j) point corresponds to the case of poles lying in the imaginary axis when discussing roots of the characteristic equation plotted in the m or p plane; a locus enclosing the (-1 + 0j) point and representing a divergent oscillation corresponds to roots of the characteristic equation lying in the right hand half of the m or p plane.

The criterion for stability may thus be stated:-

If the $KG(j\omega)$ locus is traversed from the $\omega = 0$ point in the $\omega = \infty$ direction, the system is stable provided the (-1 + 0j) point is always to the left of the locus.

This definition is valid for single loop systems and for multi-loop systems with stable inner loops.

The criterion is named after Nyquist who first stated it in relation to amplifier design and it is of course well-known in relation to the performance of feedback amplifiers.

The harmonic response locus $KG(j\omega)$ is often known as a Nyquist diagram, but it must be observed that the latter term is subject to frequent abuse and is often applied indiscriminately to polar response plots based on functions of $j\omega$ other than the $KG(j\omega)$ locus.

Further consideration of the relationship between the three basic loci is given in Appendix I.

10.3 $KG(j\omega)$ and the roots of the characteristic equation

It is convenient at this juncture to refer back to the transient response via the Laplace parameter and functions plotted on the complex plane and to consider again the roots of the characteristic equation. The major difficulty in dealing with the techniques used in synthesis is to obtain an understanding of the interrelation between different approaches to a given problem and to put a practical interpretation on some of the more abstract aspects. This is especially true when dealing with the $KG(j\omega)$ locus for the first time.

It has been shown that p is a complex number and that f(p) is also a complex number.

Thus we may write:-

$$\begin{cases} p = \alpha + j\omega \\ f(p) = u + jv \end{cases}$$

i.e. $$f(\alpha + j\omega) = u + jv$$

and the value of $u + jv$ varies continuously with changes in α and ω.

Thus $f(p)$ may be plotted (Reference 1) in terms of α and ω to form a network of constant α and constant ω lines (Figure 15).

$$\text{Since} \quad f(\alpha + j\omega) = u + jv$$

$$\text{and} \quad p = \alpha + j\omega$$

$$\frac{\partial p}{\partial \alpha} = 1 \text{ and } \frac{\partial p}{\partial \omega} = j$$

$$\frac{\partial f}{\partial p} = \frac{df}{dp}$$

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial \alpha} = \frac{df}{dp} = f'(p)$$

$$\frac{\partial f}{\partial \omega} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial \omega} = j \cdot \frac{df}{dp} = j\, f'(p)$$

Thus the spacings of the constant α and ω contours are equal at any point and intersect orthogonally.

Reference to Figure 15 shows also the plot of constant α and ω lines on the p plane which was considered in Figure 9. Again these are equally spaced and cut orthogonally, hence the $f(p)$ plane is a conformal transformation of the p plane.

We have already seen that the system response is governed by the characteristic equation:-

$$1 + KG(p) = 0$$

$$\text{or} \quad 1 + f(p) = 0.$$

The solution to this equation is the value of p which gives $f(p) = -1$. This may be determined directly in the form $(\alpha + j\omega)$ by reading from Figure 15 the values of α and ω at which the network intersects the $(-1 + 0j)$ point. The graphical work may be reduced by plotting the value of $f(\alpha + j\omega)$ for $\alpha = 0$, and constructing the complete net by the orthogonal properties of the α, ω contours.

It follows that $f(\alpha + j\omega)$ for $\alpha = 0$ is $f(j\omega)$ and that the line for $\alpha = 0$ corresponds to the harmonic response locus, when $f(j\omega) = KG(j\omega)$. If this locus passes through the $(-1 + 0j)$ point it clearly corresponds to a root of the characteristic equation with a zero real part and when it passes to the left of the $(-1 + 0j)$ point it corresponds to a positive real part to the complex roots discussed previously, and hence to a divergent oscillation. This provides additional clarification of the significance of $KG(j\omega)$.

It is possible to extend the graphical treatment given above to obtain a complete solution to the transient response to step inputs (Reference 1).

10.4 Constant dynamic magnifier and phase contours of $\frac{\theta_o}{\theta_i}(j\omega)$ in the $KG(j\omega)$ plane

Contours of constant M_M and phase for $\frac{\theta_o}{\theta_i}(j\omega)$ may be plotted in the KG plane (Figure 16).

Since M_M is the magnitude of $\frac{KG(j\omega)}{1 + KG(j\omega)}$

this may be re-arranged with x and y as Cartesian co-ordinates of the locus of $KG(j\omega)$ to give

$$y^2 + \left[x + \frac{M_M^2}{M_M^2 - 1}\right]^2 = \frac{M_M^2}{(M_M^2 - 1)^2}$$

Thus the locus of constant M_M is a circle whose centre is located on the negative real axis with co-ordinates:-

$$\left(-\frac{M_M^2}{M_M^2 - 1}, 0\right)$$

and has a radius:-

$$\frac{M_M}{M_M^2 - 1}$$

It will be noted that this locus is independent of ω, thus providing a generalised plot.

Similarly when ψ is the phase lead of output on input we may write:-

$$N = \tan \psi$$

$$(x + 0.5)^2 + \left(y - \frac{1}{2N}\right)^2 = 0.25 \left[\frac{N^2 + 1}{N^2}\right]$$

The centre of the circle is at co-ordinates:-

$$\left(-0.5, +\frac{1}{2N}\right)$$

and has radius:-

$$\frac{1}{2N}\sqrt{N^2 + 1}$$

If we consider a plot of the $KG(j\omega)$ locus with M_M and N circles superimposed it is clear that we can deduce the system response from it readily and if required plot conventional curves of M_M and ψ against frequency (Figure 17). The three curves show the effect of an increase in K:-

(a) Larger M_M values

(b) Less phase shift

(c) No change in basic locus shape

Further increase in gain would result in a violently oscillatory system.

It is seen that adjustment of gain may enable a particular M_M or ψ specification to be met, but that gain adjustment alone cannot always meet both requirements.

10.5 Adjusting K in the $G(j\omega)$ plane for a selected value of $M_{M_{max}}$

If we draw the locus of $G(j\omega)$ i.e. $K = 1$ we can draw a line at an angle δ to the real axis (Figure 18), such that $\delta = \sin^{-1} \frac{1}{M_M}$. A circle can then be drawn so that the $\sin^{-1} \frac{1}{M_M}$ line and the locus $G(j\omega)$ are tangents. This is possible since:-

$$\sin \delta = \frac{AB}{AO} = \frac{\frac{1}{K}\left(\frac{M_M}{M_M^2 - 1}\right)}{\frac{1}{K}\left(\frac{M_M^2}{M_M^2 - 1}\right)} = \frac{1}{M_M}$$

where $\frac{1}{K}$ is the scale factor of the drawing in so far as the true value of M_M is concerned.

Now $\frac{OC}{OB} = \frac{OB}{OA}$ where BC is a perpendicular to OA from B

$$\therefore \quad OC = \frac{OB^2}{OA}$$

$$= \frac{\frac{1}{K^2}\left(\frac{M_M^2}{M_M^2 - 1}\right)^2 - \frac{1}{K^2}\left(\frac{M_M}{M_M^2 - 1}\right)^2}{\frac{1}{K}\left(\frac{M_M^2}{M_M^2 - 1}\right)}$$

$$\therefore \quad OC = \frac{1}{K}$$

This gives the required value of K for the $M_{M_{max}}$ criterion selected and the locus of $KG(j\omega)$ can then be drawn as described previously. The resonant frequency can be determined as the value of ω at which the locus is a tangent to the M_M circle.

This section provides a further illustration of the value of working in terms of the $KG(j\omega)$ locus rather than the overall response $\frac{\theta_o}{\theta_i}(j\omega)$ since:-

$$\frac{\theta_o}{\theta_i}(j\omega) = \frac{KG(j\omega)}{1 + KG(j\omega)}$$

Any change in K would necessitate a complete replot of the $\frac{\theta_o}{\theta_i}(j\omega)$ locus, whereas it merely requires a change of scale for the $KG(j\omega)$ locus.

10.6 Gain and phase margin

These terms are often used in design work and may be stated in slightly different ways; one definition is:-

Gain margin The reciprocal of the $KG(j\omega)$ magnitude for a phase of $-\pi$ radians. Typical value 2.5 to 10.

Phase margin The angle between the negative real axis and the $KG(j\omega)$ vector for the frequency at which the magnitude of $KG(j\omega)$ is unity. Typical value 30° to 60°.

10.7 The inverse locus $KG^{-1}(j\omega)$

Similar properties exist for the locus of $KG^{-1}(j\omega)$ as for the locus of $KG(j\omega)$ discussed above.

Since
$$\frac{1}{\frac{\theta_o}{\theta_i}(j\omega)} = 1 + KG^{-1}(j\omega)$$

a very simple vector relation exists on the KG^{-1} plane (Figure 18) and constant M_M contours are circles centred at $(-1 + 0j)$ and radius $\frac{1}{M_M}$; constant N contours are diameters of the M_M circles making an angle $\tan^{-1} N$ with the real axis, where $N = \tan\psi$ and ψ is the phase lead of input on output.

10.8 Adjusting K in the $G^{-1}(j\omega)$ plane for a selected value of $M_{M_{max}}$

The process is similar to that used on the $G(j\omega)$ plane. The $\sin^{-1}\frac{1}{M_M}$ line is drawn for the desired $M_{M_{max}}$ where $\sin\delta = \frac{1}{M_M}$ and a circle drawn such that the locus and the $\sin^{-1}\frac{1}{M_M}$ line are tangents to it. Since the centre of the circle is located at $(-1 + 0j)$ in the KG plane it follows that if the locus is plotted to its own scale, then OA = K the desired value of K.

10.9 Modifying the $G(j\omega)$ locus

It has been shown that modification to the gain K alone cannot usually produce the required performance and the $G(j\omega)$ function itself must be re-shaped. This may be effected by modification to the system arrangement, or by introduction of "shaping circuits" which are in effect active or passive elements designed so that the introduction of their transfer functions into the loop produces phase and amplitude changes in the response locus. The addition of phase advance for example to a lagging system whose $KG(j\omega)$ locus passes through the $(-1 + 0j)$ point can move the locus into the stable region.

The general subject of reshaping the $G(j\omega)$ locus is however outside the scope of this paper.

10.10 The relative merits of the $KG(j\omega)$ and $KG^{-1}(j\omega)$ loci

It may be difficult to appreciate the relative merits of the two methods of plotting from the above, since the properties discussed appear to be very similar for either locus. Since the low frequency end of the locus lies near the origin for the inverse locus, this method of plotting is preferable for consideration of the low frequency response, but where the high frequency response is concerned this is best dealt with by the Nyquist diagram or $KG(j\omega)$ locus. The methods are thus complementary.

The inverse locus however possesses a particular advantage in dealing with multiple loop feedback systems, or systems containing dynamic elements in the feedback path. This may be illustrated by reference to the output rate feedback system considered in 4.6.1. It was shown that the basic equation was of the form:-

$$\theta'(t) = \theta(t) - Q(D)\theta_o(t)$$

In the absence of the additional feedback term we may write:-

$$\theta_o(t) = KG_1(D)\theta(t)$$

Hence with the feedback term:-

$$\theta_o(t) = KG_1(D)\ \theta(t) - Q(D)\theta_o(t)$$

$$\theta_o(t) = \frac{KG_1(D)}{1 + Q(D)\ KG_1(D)} \cdot \theta(t)$$

or

$$\frac{\theta_o}{\theta}(j\omega) = \frac{KG_1(j\omega)}{1 + Q(j\omega)KG_1(j\omega)}$$

Thus with the harmonic response function the locus must be recalculated for any change in the feedback term $Q(j\omega)$.

The inverse locus however is:-

$$\frac{\theta}{\theta_o}(j\omega) = KG_1^{-1}(j\omega) + Q(j\omega)$$

which enables the additional loops to be included by simple vector addition of the feedback term to the inverse locus of the system which is to be modified.

11.0 The logarithmic or asymptotic form of locus

The asymptotic form of locus was proposed over twenty years ago and has been intensively developed in the field of communications engineering during recent years.

The technique comprises plotting the magnitude and phase of the transfer function separately against the frequency in logarithmic coordinates: some surprise may be occasioned by this since it appears to be a reversion to techniques considered earlier and rejected as unsuitable for synthesis. The significance lies however in the use of logarithmic coordinates.

The logarithm of the magnitude of $KG(ju)$ at any point may be written:-

$$\log KG(ju) = \log K + \log G(ju)$$

The phase or angle of $KG(ju)$ is:-

$$\text{Ang } KG(ju) = \text{Ang } G(ju)$$

For a complex system:-

$$KG(ju) = \frac{K_1 G_1(ju) \cdot KG_2(ju)}{K_3 G_3(ju)} \text{ etc.}$$

At a particular value of $KG(ju)$ we may write for the amplitude:-

$$\log KG(ju) = \log K_1 + \log K_2 - \log K_3 + \log G_1(ju) + \log G_2(ju) - \log G_3(ju)$$

Thus former multiplication and division become addition and subtraction and change in gain merely requires the addition or subtraction of a constant.

Since this approach is allied to general techniques of a similar nature in communications engineering, the same units are employed i.e. decibels. These originated largely as the result of the human ear having a basically logarithmic response. Thus successive increases in sound intensity of say ten times would create the impression of equal increments to the human ear, rather than equal multiplication.

It follows that if a power level of P_2 is established, greater than P_1, the increase in level of P_2 over P_1 is $\log \frac{P_2}{P_1}$ relative to a datum at P_1. Since it is conventional to work in $\log_{10}$ we can write:-

$$\text{Change in power level} = \log_{10} \frac{P_2}{P_1} \text{ in bels}$$

$$\equiv 10 \log_{10} \frac{P_2}{P_1} \text{ in decibels.}$$

Now if the input and output impedances are equal and say R:-

$$\left.\begin{aligned} P_2 &= \frac{V_o^2}{R} \\ P_1 &= \frac{V_i^2}{R} \end{aligned}\right\}$$

Where V_o and V_i are the output and input voltages or amplitudes

Hence change in voltage or

$$\text{signal amplitude level} = 20 \log_{10} \left(\frac{V_o}{V_i}\right) \text{ decibels.}$$

When $V_o = 2\,V_i$ the change in level is $20 \cdot 0.301 \simeq 6$ decibels (db) and it will be noted that a further doubling of the output would result in a further increase of 6 db making 12 db in all from the datum level.

It is not uncommon for the subject to be discussed on a similar basis in respect of frequency, in that octaves are used to describe change in frequency level or pitch; thus doubling the frequency represents an increase in frequency level of one octave.

Hence change in frequency level relative to a datum at f_1 $= \log_2\left(\frac{f_2}{f_1}\right)$ octaves where f_2 and f_1 are the respective frequencies.

It is convenient to abbreviate the expression:-

$$20 \log_{10} KG(ju) = 20 \log_{10} K + 20 \log_{10} G(ju)$$

to $\mathrm{Lm}\, KG(ju)$ where the units are decibels. The phase relation $\mathrm{Ang}\, G(ju)$ is usually in degrees.

As an example, we may now consider the transfer function:-

$$KG(ju) = \frac{1}{1 + ju}$$

$$\mathrm{Lm}\, KG(ju) = \mathrm{Lm}\, K + \mathrm{Lm}\, (1 + ju)^{-1}$$

$$= \mathrm{Lm}\, (1 + ju)^{-1}$$

The amplitude and phase logarithmic plots are given in Figure 19. The amplitude plot exhibits interesting properties which must be considered.

When	$u \to \infty$	$\mathrm{Lm}\, G(ju) \to -\infty$
	$u \to 0$	$\mathrm{Lm}\, G(ju) \to 0$
	$u = 0.5$	$\mathrm{Lm}\, G(ju) = -1$ db
	$u = 1.0$	$\mathrm{Lm}\, G(ju) = -3$ db
	$u = 2.0$	$\mathrm{Lm}\, G(ju) = -7$ db
	$u >>> 1$	$\mathrm{Lm}\, G(ju) \simeq \log\, (ju)^{-1}$

Thus at the higher frequencies the amplitude falls by 6 db for each doubling in frequency; this is often quoted as the amplitude falling 6 db per octave.

Thus two asymptotes describe the function, one at a constant level of 0 db and the other starting at 0 db, $u = 1$ and falling at 6 db per octave; if desired the three corrective points calculated above may be used to remove the error in the region $u = 1$.

The point of intersection of the two asymptotes is known as the break point, which here is at $u = 1$; this is known as the corner frequency and occurs at the value of u which makes the real and imaginary parts of the function numerically equal.

It will be noted that variation in gain factor K merely changes the reference level of the whole plot.

The other basic factors of the transfer function can be treated in a similar manner and the general method forms a powerful tool in synthesis, complementary to the vector loci techniques.

12.0 General consideration of transfer function loci shapes

We have seen that the characteristic equation may be factorised to give:-

$$(p - \gamma_1)(p - \gamma_2) \dots (p - \gamma_k) \dots (p - \gamma_{n-1})(p - \gamma_n) = 0$$

and that the roots may be real, imaginary, or in complex pairs, and in general will have the basic forms:-

$$(p + 0)$$

$$(p\tau + 1)$$

$$(p^2 + 2\zeta\,\omega_n\,p + \omega_n^2)$$

By making a substitution of the form $u = f(p)$ we obtain basic forms:-

$$ju$$

$$(ju + 1)$$

$$\left[(ju)^2 + 2\zeta\,(ju) + 1\right]$$

and in general the transfer function will consist of factors of the form:-

$$(ju)^{\pm 1}$$

$$(ju + 1)^{\pm 1}$$

$$\left[(ju)^2 + 2\zeta\,(ju) + 1\right]^{\pm 1}$$

$$(ju\lambda + 1)^{\pm 1}$$

$$\left[(ju\lambda)^2 + 2\zeta\,(ju\lambda) + 1\right]^{\pm 1}$$

Where $KG(ju)$ has the form $K \cdot \dfrac{A(ju) \cdot B(ju)}{C(ju) \cdot D(ju)}$ etc.

One of the most important properties of the transfer function locus lies in the fact that an individual skilled in the art can by visual inspection alone, obtain a considerable amount of data from it relating to the general properties of the system or element.

In Figure 20 a collection of the more common and useful loci are tabled, together with a diagram of the locus shape for each. Provided the transfer function locus is available for a system or element the latter may generally be

classified even if the exact mathematical function is unknown. Reference 4 may be consulted for a comprehensive consideration of harmonic loci.

Amongst the functions tabled are the basic first and second order of control functions in (1) and (2), the ideal derivative term controller function in (6) and the ideal integral terms controller function in (3). Since the latter require infinite gain at infinite and zero frequency respectively, they are impracticable for general use and would in fact be undesirable in most systems due to the presence of noise and random signals which require the operating frequency range to be limited to the minimum acceptable band width. The functions in (7) and (4) are more representative of realistic derivative and integral term responses.

Typical shaping circuits for obtaining various input-output element characteristics are shown in Figure 21.

An important operator considered in 4.5.1 was the finite delay and it was shown to have the transfer function form $\frac{1}{e^{TD}}$ or $\frac{1}{e^{ju}}$. The destabilising effect was discussed and may be illustrated by consideration of the effect of this operator on the harmonic response locus.

Now $e^{-ju} = \cos u - j \sin u$

and represents a rotation of the input vector by an angle $-u$, since the amplitude is unchanged. Thus at any point on the harmonic response locus frequency u_k, the delay operator rotates the vector and causes it to lag by u angular units (Figure 22).

13.0 Conclusions

The properties of elementary control systems have been considered, the basic terminology described and the conception of order of control introduced.

The presence and effect of lag terms in a practical system was noted, particularly that of a finite time delay and the conception of a controller whose response could be adjusted to give the desired response was introduced.

The importance of the properties of the roots of the characteristic equation were emphasised and the location of them in the m plane noted in relation to stability and system response. The damping term in a simple system was shown to result in three basic forms of the roots of the characteristic equation, leading to underdamped, critically damped and overdamped responses.

Initially step functions of input were employed, but consideration of a harmonic input illustrated the amplitude and phase properties of a system with variation in input frequency and the limitation of response with frequency dictated by the constants of the system. The correlation between transient and harmonic response was emphasised and the ability to transform from one domain to the other by various techniques noted. It was shown that consideration of the harmonic response after the decay of the initial condition terms, reduced the response function to the particular integral solution of the differential equation and use of complex numbers indicated that the complex frequency parameter $(j\omega)$ could be substituted for D or m to give the harmonic response on a vector basis.

The conception of an element and loop transfer function was introduced relating output to input for the element and output to error for the loop, which comprised complex numbers completely describing the properties of the element or system under dynamic conditions. A brief note was recorded on the use of the Laplace transformation method, which is much favoured in the U.S.A. and is a powerful method for the solution of complex systems.

The crux of the problem was reached in considering analysis and synthesis, the transient approach leading to analysis of system behaviour but being relatively intractable for determining the optimum system arrangement to meet the desired performance specification i.e. the process of synthesis.

Achievement of a satisfactory input-output response is obviously the goal of the designer, but it was shown that achievement of the desired response by synthesis was powerfully aided by the frequency response properties of the output-error function to a harmonic input i.e. the loop transfer function KG expressed as a function of $(j\omega)$ or of a non-dimensional frequency parameter (ju) and known as a Nyquist diagram. Various graphical constructions were considered leading to establishment of the stability and gain performance with frequency and of the relation between plots in the $KG(j\omega)$ plane and the roots of the characteristic equation. The relative merits of the harmonic response locus $KG(j\omega)$ and the inverse locus $KG^{-1}(j\omega)$ were considered, the latter being of advantage when dealing with multiple loops and dynamic elements in the feedback path. The logarithmic locus was mentioned briefly as a further aid to synthesis and finally a number of the more commonly used elements and their transfer function loci were tabled. In this connection the de-stabilising effect of a finite time delay was illustrated by demonstrating its effect as an operator on the harmonic response locus.

Consideration of the engine response rate programme may raise doubts as to the necessity for such an extensive consideration of control system theory, particularly techniques of synthesis which lie rather outside the scope of the programme. Much of the currently available literature however originated in the U.S.A. where the fullest exploitation of servo-system technology in the gas turbine response rate field has been attempted. Whilst this is unquestionably a valid approach, it has resulted in some of the basic papers which concern fundamentally simple systems being stated in terms which to a non-specialist render them incomprehensible, but which would be readily understood if expressed in a more conventional manner. For this reason Part I of the paper has been extended to cover most of the terminology used in current papers.

It may be concluded from consideration of the wealth of servo-system technique already available, that provided the engine element functions can be defined in an acceptable manner and the system lags and non-linearities established, solution of the engine problem by established means should be possible (Reference 3).

Where however unpredictable changes such as variable lags or general inconsistency are present, or the system cannot be linearised by acceptable means, it is inevitable that the problem will become exceedingly complex.

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Table I

Performance of basic displacement - displacement control systems

Steady state error	Class or order of control	Description	Basic dynamic response (No lags, friction, etc.)	Example of use	Governing equation
Position error for position input	0	Displacement error or proportional control	Instantaneous	Voltage regulator	$\theta_o(t) = K\theta(t)$ $\theta_o(t) = \frac{K}{1+K}\theta_i(t)$ $\theta(t) = \frac{1}{1+K}\theta_i(t)$
No position error for position input. Position error for velocity input.	1	Zero displacement error or velocity controlled	Exponential	Low performance remote position control	$D\theta_o(t) = K\theta(t)$ $\theta_o(t) = \frac{K}{D+K}\theta_i(t)$ $\theta = \frac{D}{D+K}\theta_i(t)$
No position error for position or velocity input. Position error for acceleration input.	2	Zero velocity error or acceleration controlled	Continuous oscillation	Electric remote position control	$D^2\theta_o(t) = K\theta(t)$ $\theta_o(t) = \frac{K}{D^2+K}\theta_i(t)$ $\theta = \frac{D^2}{D^2+K}\theta_i(t)$
No position error for position, velocity, or acceleration input.	3	Zero acceleration error control	Divergent oscillation	-	$D^3\theta_o(t) = K\theta(t)$ $\theta_o(t) = \frac{K}{D^3+K}\theta_i(t)$ $\theta = \frac{D^3}{D^3+K}\theta_i(t)$

Table II

Response of zero displacement error system to various disturbing functions

Form of disturbance	$\zeta < 1$	$\zeta = 1$	$\zeta > 1$
Step displacement of input θ_i	$\frac{\theta(t)}{\theta_i} = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2} \cdot \omega_n t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $\theta_{ss} = 0$	$\frac{\theta(t)}{\theta_i} = e^{-\omega_n t}(1 + \omega_n t)$	$\frac{\theta(t)}{\theta_i} = e^{-\zeta\omega_n t} \left[\cosh \sqrt{\zeta^2 - 1} \cdot \omega_n t + \frac{\zeta}{\sqrt{\zeta^2-1}} \sinh \sqrt{\zeta^2 - 1} \cdot \omega_n t\right]$
Step velocity of input $\omega_i = \frac{d\theta_i}{dt}$	$\frac{\theta(t)}{\theta_{ss}} = 1 - \frac{e^{-\zeta\omega_n t}}{2\zeta\sqrt{1-\zeta^2}} \cdot \sin(\sqrt{1-\zeta^2} \cdot \omega_n t + \phi)$ $\phi = \tan^{-1} \frac{2\zeta\sqrt{1-\zeta^2}}{(2\zeta^2-1)}$ $\theta_{ss} = \frac{2\zeta\omega_i}{\omega_n}$	$\frac{\theta(t)}{\theta_{ss}} = 1 - e^{-\omega_n t}\left(1 + \frac{\omega_n t}{2}\right)$	$\frac{\theta(t)}{\theta_{ss}} = 1 - e^{-\zeta\omega_n t} \left[\cosh \sqrt{\zeta^2 - 1} \cdot \omega_n t + \frac{2\zeta^2 - 1}{2\zeta\sqrt{\zeta^2-1}} \sinh \sqrt{\zeta^2 - 1} \cdot \omega_n t\right]$
Step application of load torque Γ_L	$\frac{\theta(t)}{\theta_{ss}} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2} \cdot \omega_n t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $\theta_{ss} = \frac{\Gamma_L}{J\omega_n^2}$	$\frac{\theta(t)}{\theta_{ss}} = 1 - e^{-\omega_n t}(1 + \omega_n t)$	$\frac{\theta(t)}{\theta_{ss}} = 1 - e^{-\zeta\omega_n t} \left[\cosh \sqrt{\zeta^2 - 1} \cdot \omega_n t + \frac{\zeta}{\sqrt{\zeta^2-1}} \sinh \sqrt{\zeta^2 - 1} \cdot \omega_n t\right]$

$$\omega_n = \sqrt{\frac{K}{J}} \qquad \zeta = \frac{f}{2\sqrt{JK}} \qquad 2\zeta\omega_n = \frac{f}{J}$$

Table III

Some basic Laplace Transform pairs

Description of function		$f(t), 0 \leq t$	$f(p)$
Step or unit function		$f(t) = 1$ or $u(t)$	$\frac{1}{p}$
Step function with finite time delay		$f(t) = u(t - T)$ step delayed by time = T	$\frac{1}{p} \cdot e^{-pT}$
Impulse function		$f(t) = u(t) - u(t - T), T > 0$ impulse duration T commencing at $t = 0$	$\frac{1}{p}(1 - e^{-pT})$
Impulse function with finite time delay		$f(t) = u(t - T_1) - u(t - T_2), T_2 > T_1$ impulse duration $(T_2 - T_1)$ delayed by time = T_1	$\frac{1}{p}\left[e^{-pT_1} - e^{-pT_2}\right]$
Simple exponential	Decay	$f(t) = e^{-\omega t}$	$\frac{1}{p + \omega}$
	Delay	$f(t) = \frac{1}{\tau} e^{-t/\tau}$	$\frac{1}{1 + p\tau}$
Quadratic factor or complex exponential delay	Underdamped	$\zeta < 1, f(t) = \frac{1}{\omega_n\sqrt{1 - \zeta^2}} \cdot e^{-\zeta\omega_n t} \sin\sqrt{1 - \zeta^2} \cdot \omega_n t$	$\frac{1}{p^2 + 2\zeta\omega_n p + \omega_n^2}$
	Critically damped	$\zeta = 1, f(t) = te^{-\omega_n t}$	
	Overdamped	$\zeta > 1, f(t) = \frac{1}{\omega_n\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n t} \sinh\sqrt{\zeta^2 - 1} \cdot \omega_n t$	
Harmonic function	Sine	$f(t) = \sin \omega t$	$\frac{\omega}{p^2 + \omega^2}$
	Cosine	$f(t) = \cos \omega t$	$\frac{p}{p^2 + \omega^2}$

APPENDIX I

Further consideration of the relationship between the loci $KG(j\omega)$, $KG^{-1}(j\omega)$ and $\frac{\theta_o}{\theta_i}(j\omega)$

Further consideration of the basic vector diagram may help to clarify the relationship between the three primary loci $KG(j\omega)$, $KG^{-1}(j\omega)$ and $\frac{\theta_o}{\theta_i}(j\omega)$ i.e. the Nyquist diagram, the inverse locus and the harmonic locus.

D. B. Welbourn has drawn the attention of the author to an illustration based on a particular example. Consider for example a damped simple spring-mass system:-

$$(T_2^2 D^2 + T_1 D + 1)\,\theta_o(t) = \theta_i(t)$$

$$\text{where} \quad T_1 = 2\zeta\, T_2$$

When the system is subjected to harmonic input:-

$$(T_2^2 D^2 + T_1 D + 1)\,\theta_o(t) = A\cos\omega t$$

the solution for the output will have the form:-

$$\theta_o(t) = B\cos(\omega t - \psi)$$

By writing $\qquad D \equiv j\omega$

$$\left[T_2^2 (j\omega)^2 + T_1(j\omega) + 1\right]\theta_o(t) = \theta_i(t) = A\cos\omega t$$

It is now possible to plot a vector diagram representing this equation, noting that the operator j rotates the vector through 90° and that j^2 rotates the vector through 180°. This is given in Figure 23(a). UV, VW and WX represent the magnitudes of θ_o, $\omega T_1\theta_o$ and $-\omega^2 T_2^2\theta_o$ respectively. These vectors sum to give UX denoting the vector θ_i. It follows that VX represents the difference vector $\theta_i - \theta_o$ i.e. the error θ. If the diagram is scaled to make UV = 1, VX represents the vector defining the locus $\frac{\theta}{\theta_o}(j\omega)$ i.e. the inverse locus.

If the diagram is now rearranged (Figure 23 (b)) with the origin at U and UX = 1, vector UV defines the locus $\frac{\theta_o}{\theta_i}(j\omega)$, or in brief the harmonic locus.

Finally by taking the origin at V and making VX = 1, vector VU defines the locus $\frac{\theta_o}{\theta}(j\omega)$, i.e. the Nyquist diagram.

APPENDIX II

List of symbols

A	=	a constant
	=	a constant coefficient in a differential equation
	=	a Fourier coefficient e.g. $A_n^2 = a_n^2 + b_n^2$
a	=	a constant
	=	a Fourier coefficient
	=	the real part of a complex root pair $(a \pm jb)$
B	=	a constant
b	=	a constant
	=	a Fourier coefficient
	=	the imaginary part of a complex root pair $(a \pm jb)$
C	=	a constant
	=	the real part of $(C + jD)$
c	=	a constant of integration
D	=	the differential operator $\equiv \frac{d}{dt}$
	=	the imaginary part of $(C + jD)$
d	=	the dimensionless frequency ratio $\frac{\omega}{\omega_n}$
e	=	the natural base of logarithms = 2.7183
	=	instantaneous e.m.f.
f	=	the coefficient of viscous friction
	=	a function e.g. of time $f(t)$
	=	a periodic frequency in cycles per unit time
f_c	=	the coefficient of viscous friction for critical damping = $2\sqrt{JK}$
f'	=	the first derivative of a function e.g. $\frac{df}{dt}$
$f(0+)$	=	the value of a function as the zero is approached from the positive direction
G	=	the frequency dependent part of a transfer function KG( )
g	=	the acceleration due to gravity
i	=	instantaneous electrical current

APPENDIX II (cont'd)

J = polar moment of inertia

j = $\sqrt{-1}$

K
- = a constant
- = a stiffness in load per deflection units
- = a gain factor e.g. the frequency invariant part of a transfer function KG( )

K_v = the velocity constant $\frac{\omega_n}{2\zeta}$

k
- = a constant
- = a stiffness in load per deflection units

L = electrical inductance

ℓ = the coefficient of the first derivative term in a generalised torque equation $\Gamma = \left(K + \ell \frac{d\theta}{dt} + m \frac{d^2\theta}{dt^2}\right)$

M_M = dynamic magnifier i.e. ratio of dynamic amplitude to equivalent static deflection

m
- = a root of the auxiliary equation
- = the coefficient of the second derivative term in a generalised torque equation $\Gamma = \left(K + \ell \frac{d\theta}{dt} + m \frac{d^2\theta}{dt^2}\right)$

N = the phase parameter $\tan \psi$

P = electrical power

p = the Laplacian parameter

$f(p)$ = the Laplace transform of $f(t)$ i.e. $\mathcal{L}\left[f(t)\right]$

Q = a function of an operator or complex variable e.g. $Q(D)$, $Q(j\omega)$

R = electrical resistance

s = the Laplacian parameter in American literature

T
- = a finite time delay
- = the periodic time $\frac{2\pi}{\omega}$

t = instantaneous time

u
- = a dimensionless frequency parameter
- = the real part of the complex number representing $f(p)$, i.e. $(u + jv)$

V = electrical voltage

v = the imaginary part of the complex number representing $f(p)$, i.e. $(u + jv)$

APPENDIX II (cont'd)

$\left.\begin{array}{l}KG(D)\\KG(p)\\KG(j\omega)\\KG(ju)\end{array}\right\}$ = a transfer function

$KG^{-1}(\ \ )$ = the inverse form of a transfer function

α $\left\{\begin{array}{l}= \text{the real part of a complex root e.g. } (\alpha_k + j\,\beta_k)\\ = \text{the real part of the complex number } p, \text{ i.e. } (\alpha + j\omega)\end{array}\right.$

β = the imaginary part of a complex root e.g. $(\alpha_k + j\,\beta_k)$

Γ = torque

γ = a factored root of the auxiliary equation

δ = $\sin^{-1}\dfrac{1}{M_M}$

ε $\left\{\begin{array}{l}= \text{a signal originating at a point in a closed loop system}\\ = \text{a phase angle}\end{array}\right.$

ζ = the ratio of actual damping coefficient to the coefficient for critical damping $= \dfrac{f}{f_c} = \dfrac{f}{2\sqrt{JK}}$

θ = angular displacement

μ = the gain of an amplifier

τ = a time constant

ϕ = a phase angle

ψ = a phase angle e.g. in a Fourier series $\psi_n = \tan^{-1}\dfrac{b_n}{a_n}$

ω = frequency

ω_n = the natural frequency of undamped oscillation = $\sqrt{\dfrac{K}{J}}$

$\omega_n\sqrt{1-\zeta^2}$ = the natural frequency of damped oscillation

$\mathcal{L}$ = denotes application of the Laplace transform

Suffices

c = critical in relation to the damping coefficient f

f = electrical field

i = input

i_{ss} = input at steady state

APPENDIX II (cont'd)

k	=	the k^{th} term
L	=	load
o	=	output
o_{ss}	=	output at steady state
M	=	motor
n	=	the n^{th} term
n	=	natural frequency in relation to frequency ω

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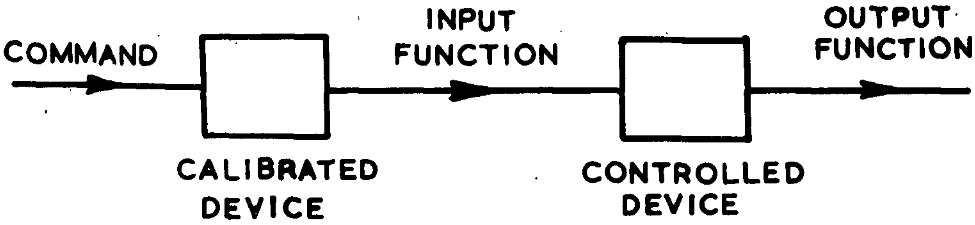
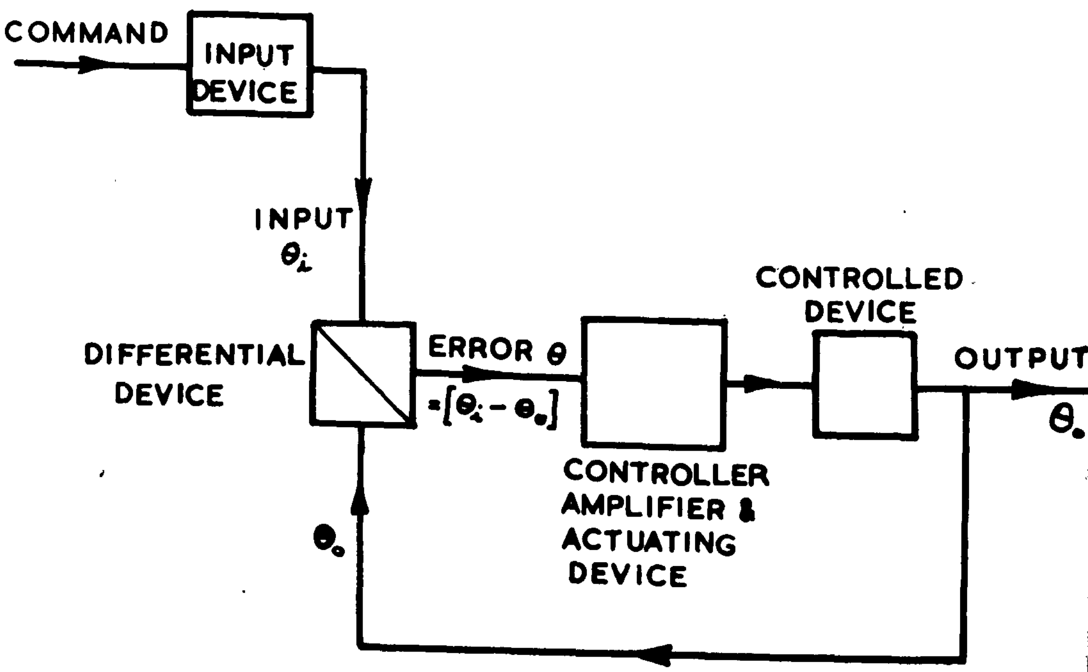
FIG. 1.



OPEN CHAIN SYSTEM.



CLOSED LOOP SYSTEM.

FIG. 2.

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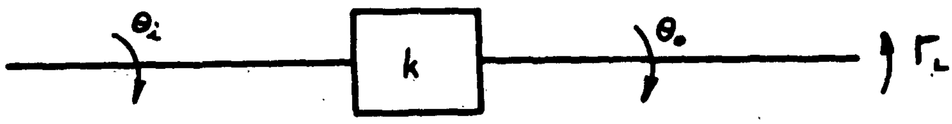
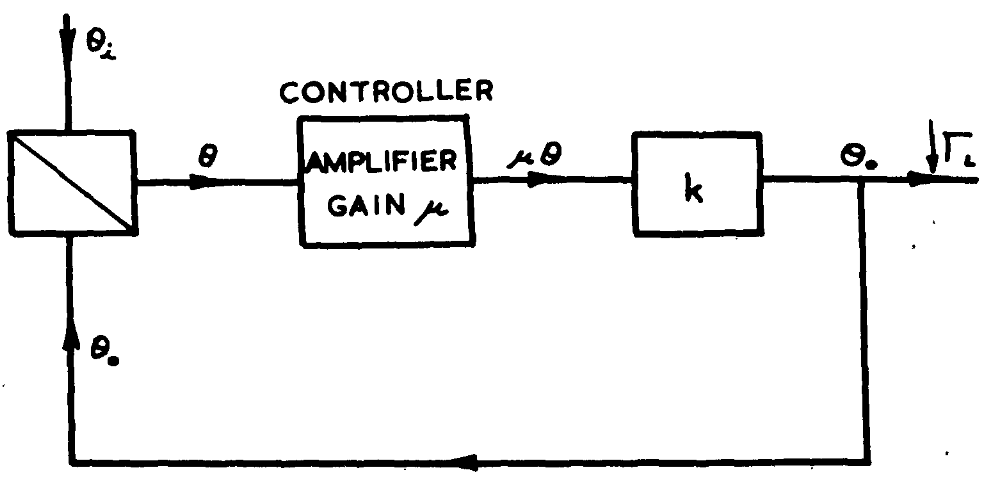


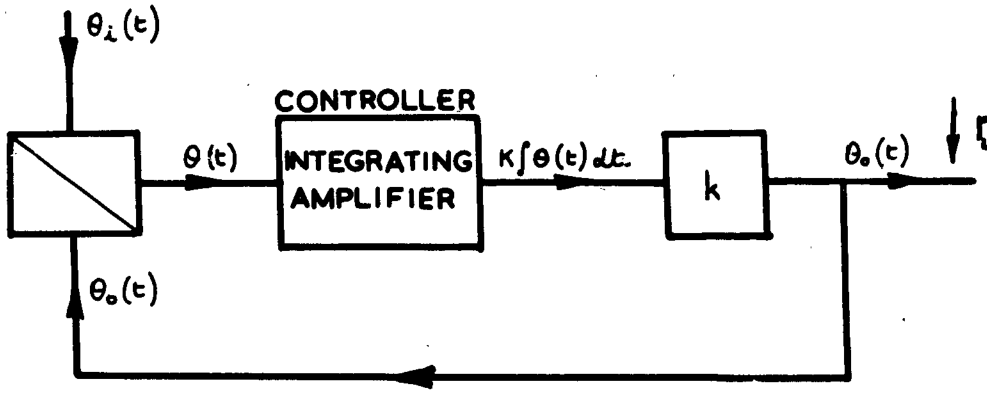
CLASS 'O' OPEN CHAIN SYSTEM.



CLASS 'O' CLOSED LOOP SYSTEM.

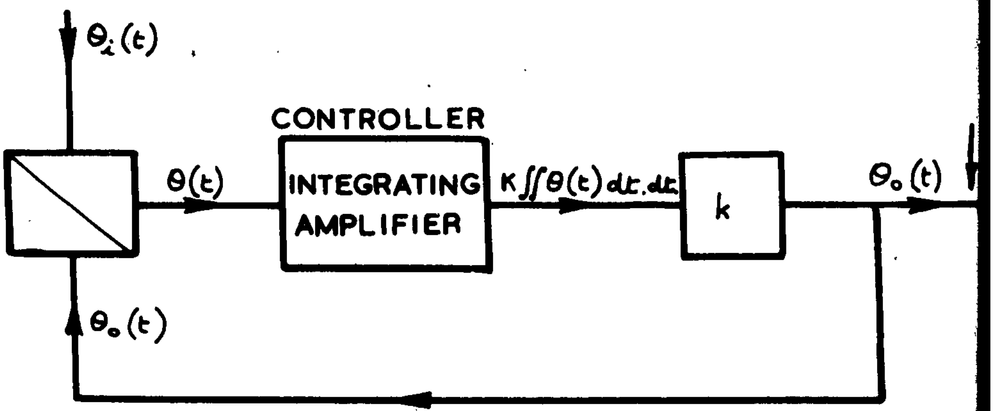
SK 26330

FIG. 3.



BASIC CLASS '1'

CLOSED LOOP SYSTEM.



BASIC CLASS '2'

CLOSED LOOP SYSTEM.

FIG. 4.

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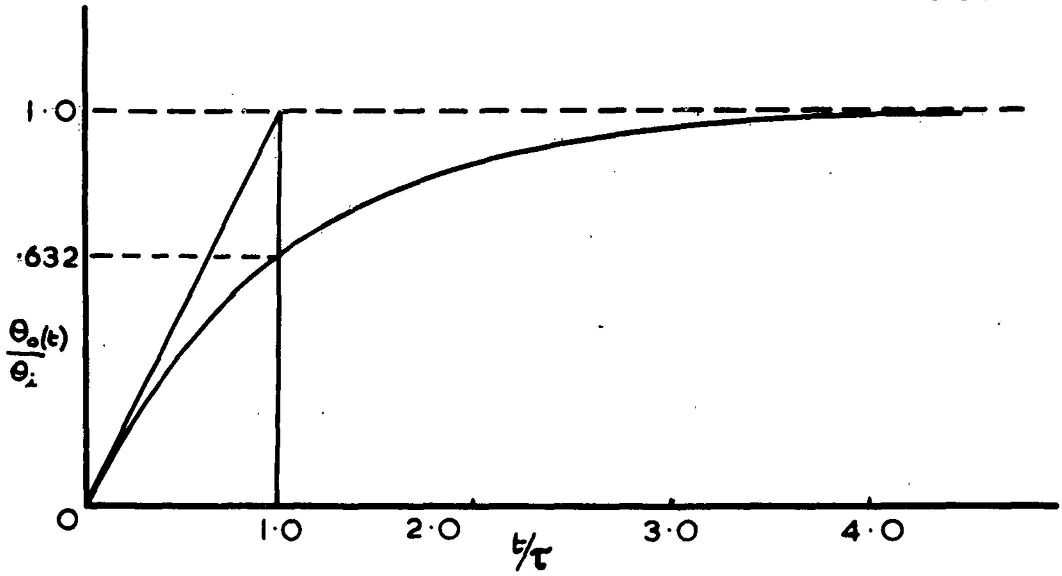
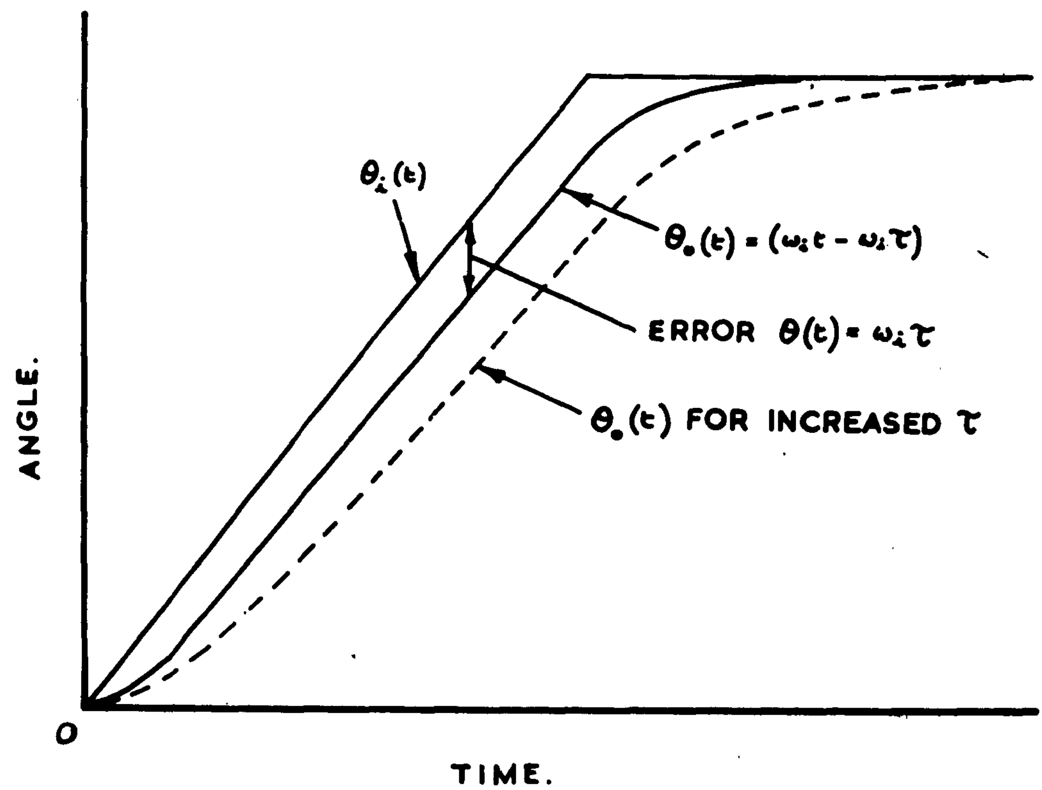


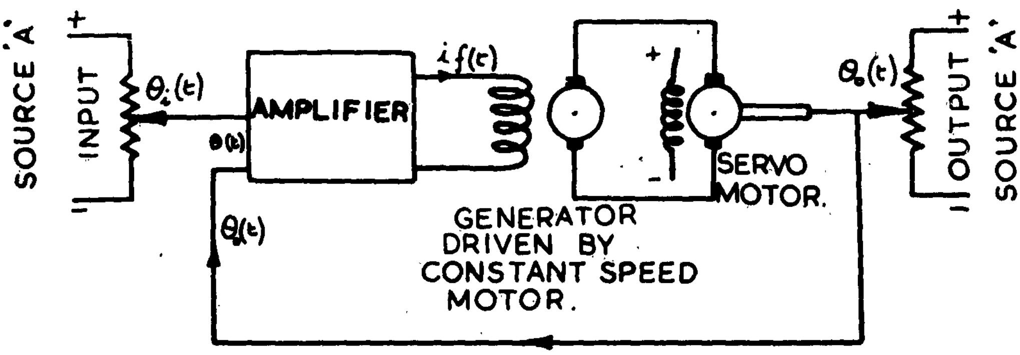
RESPONSE OF CLASS 'I' SYSTEM TO STEP INPUT DISPLACEMENT θ_i



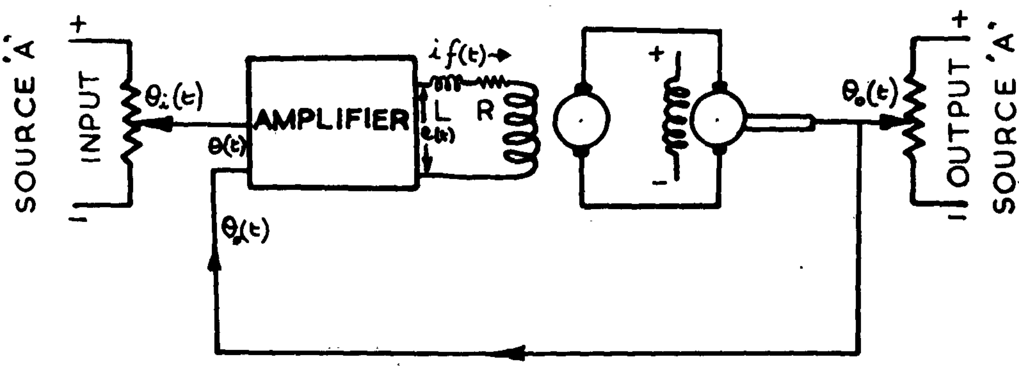
RESPONSE OF CLASS 'I' SYSTEM TO STEP INPUT VELOCITY ω_i

SK 26332

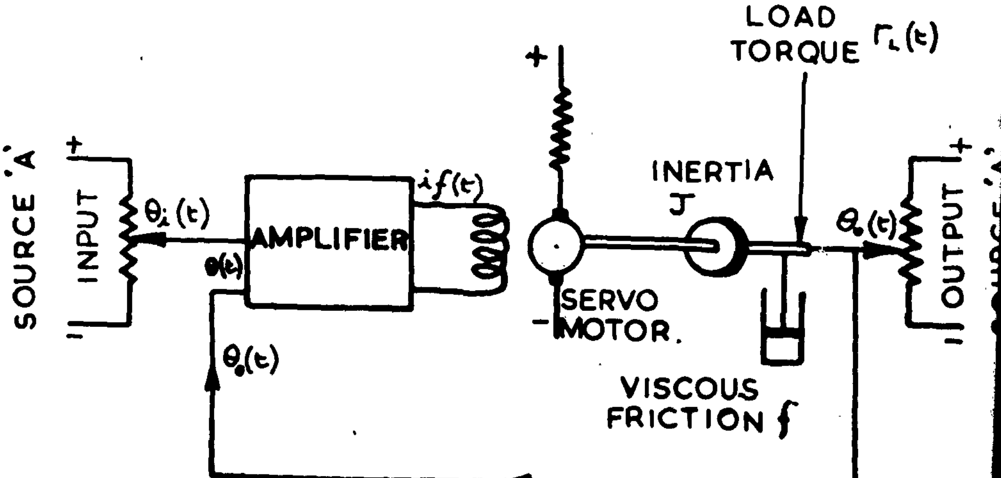
FIG. 5.



REMOTE POSITION CONTROL WITHOUT LAG



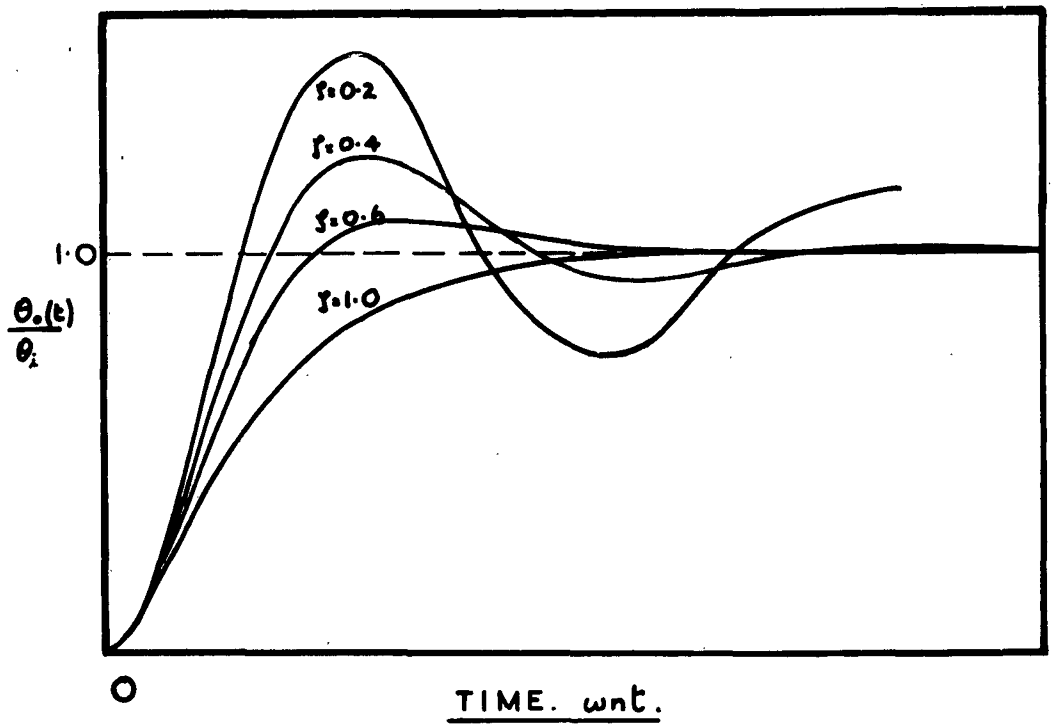
REMOTE POSITION CONTROL WITH FIELD LAG



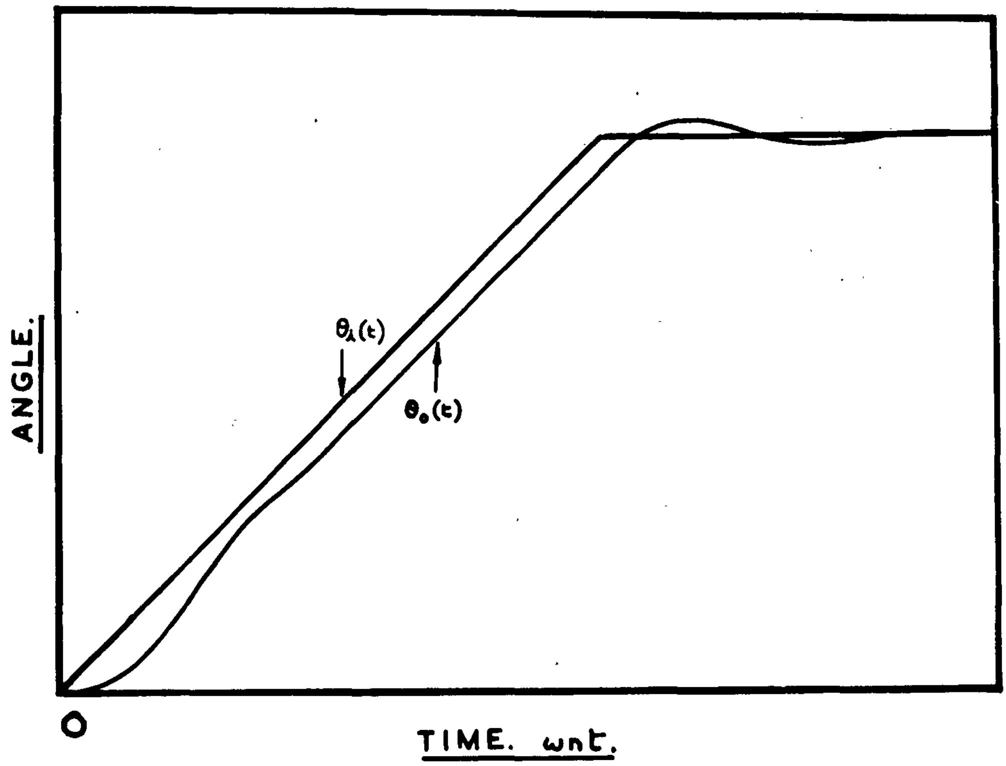
REMOTE POSITION CONTROL WITH INERTIA AND VISCOUS FRICTION.

FIG. 6.

SK 26333



RESPONSE TO STEP INPUT DISPLACEMENT θ_i.

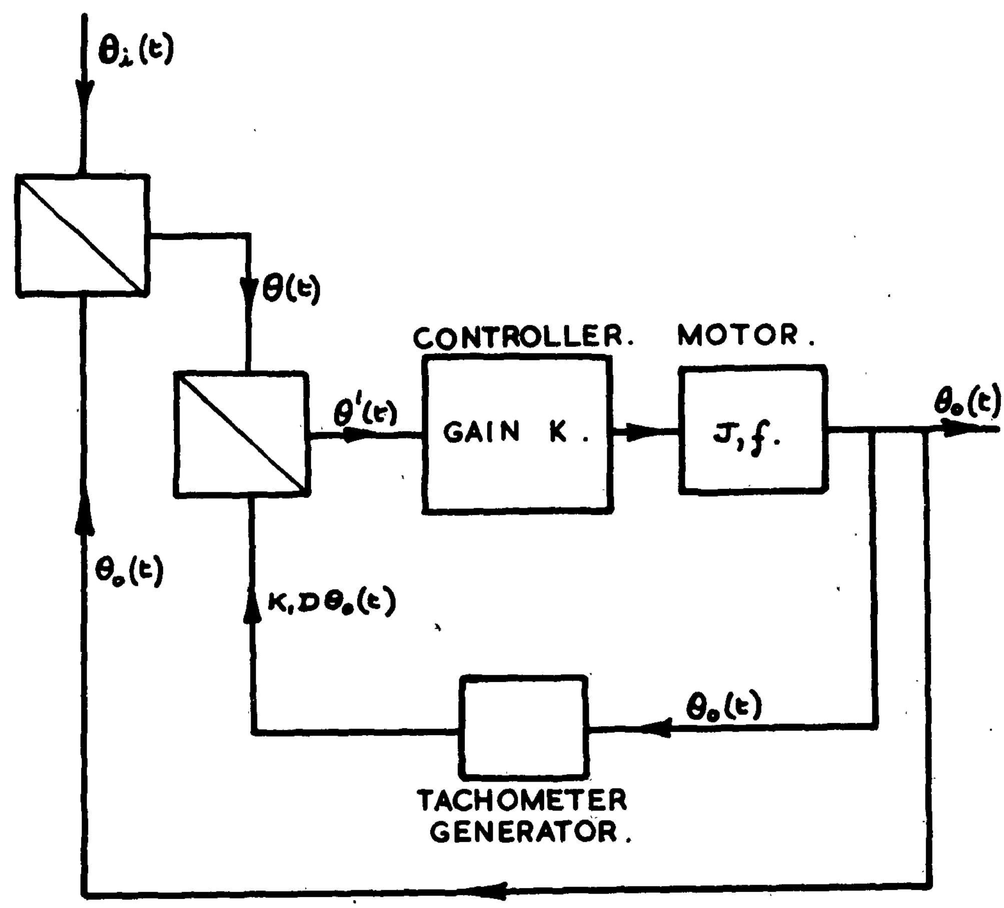


RESPONSE TO STEP INPUT VELOCITY ω_i.

ZERO DISPLACEMENT ERROR SYSTEM WITH VISCOUS FRICTION AND INERTIA.

SK 26334.

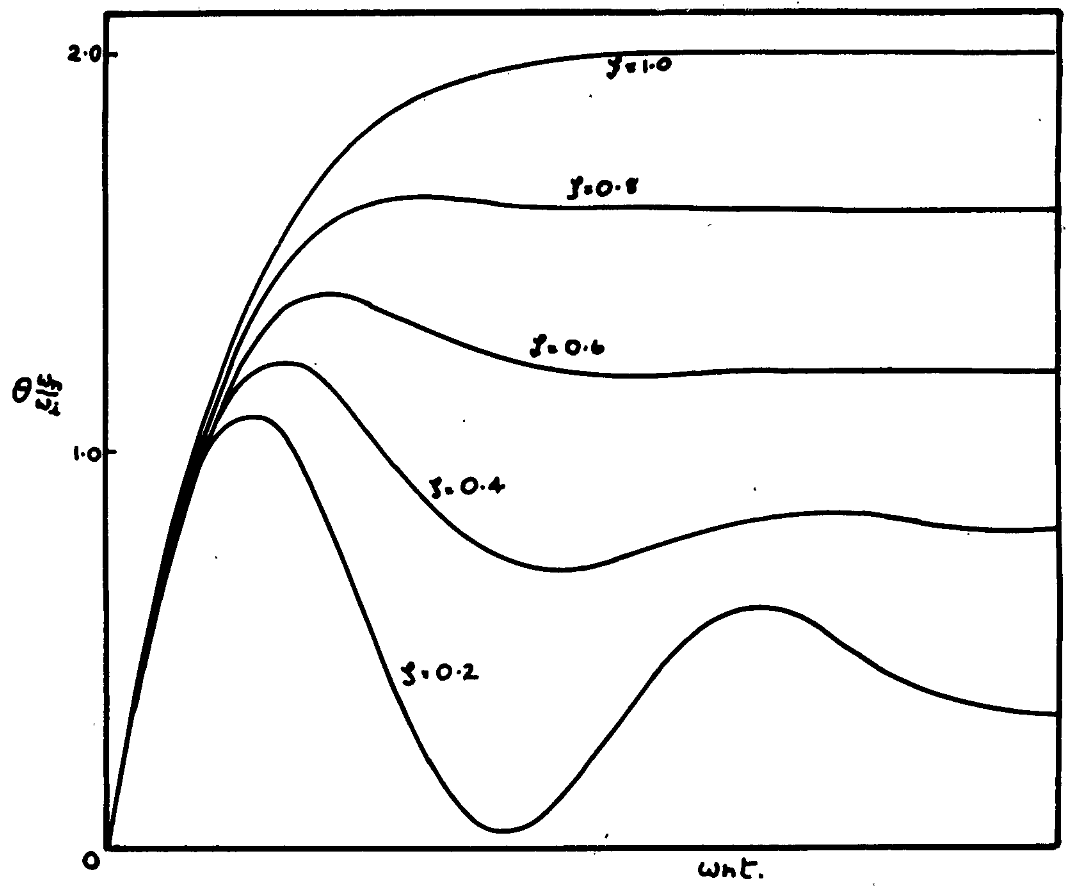
FIG. 7.



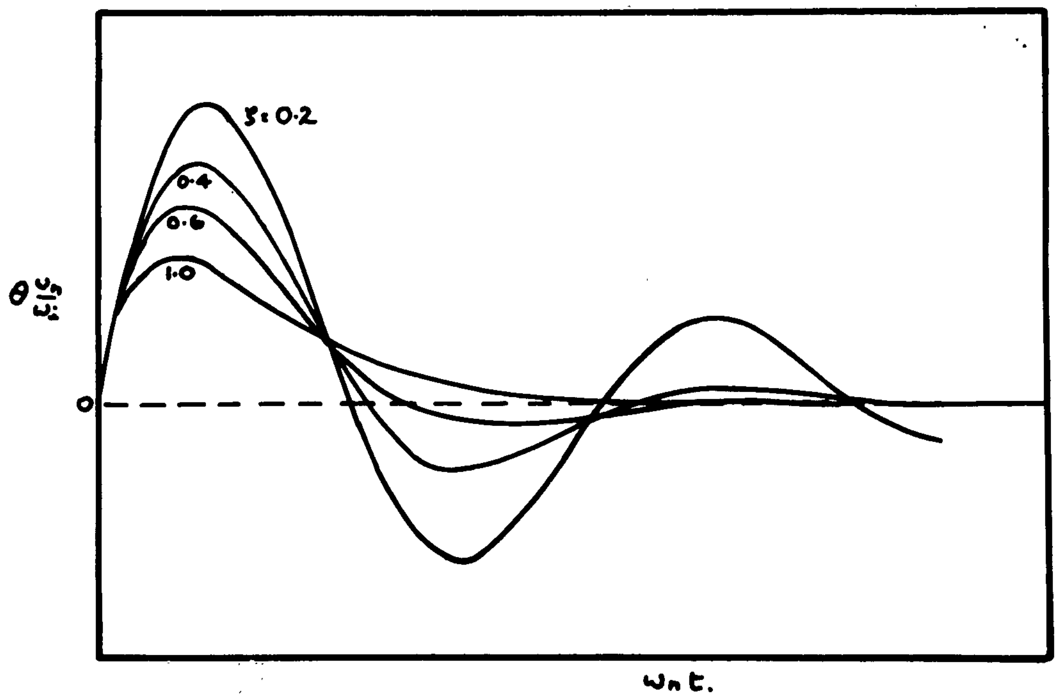
ZERO DISPLACEMENT ERROR SYSTEM WITH OUTPUT RATE FEEDBACK.

FIG. 8.

SK 26335.



VISCOUS FRICTION DAMPING.

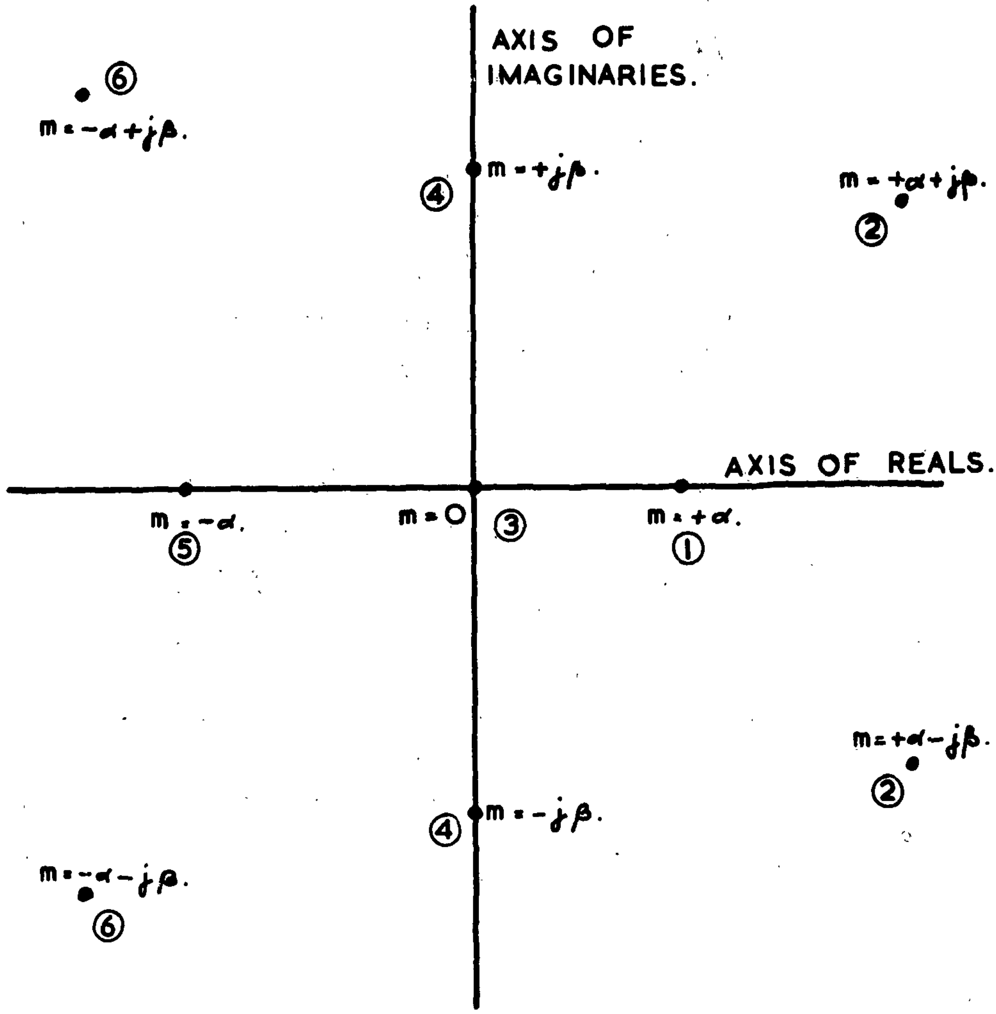


ERROR RATE DAMPING.

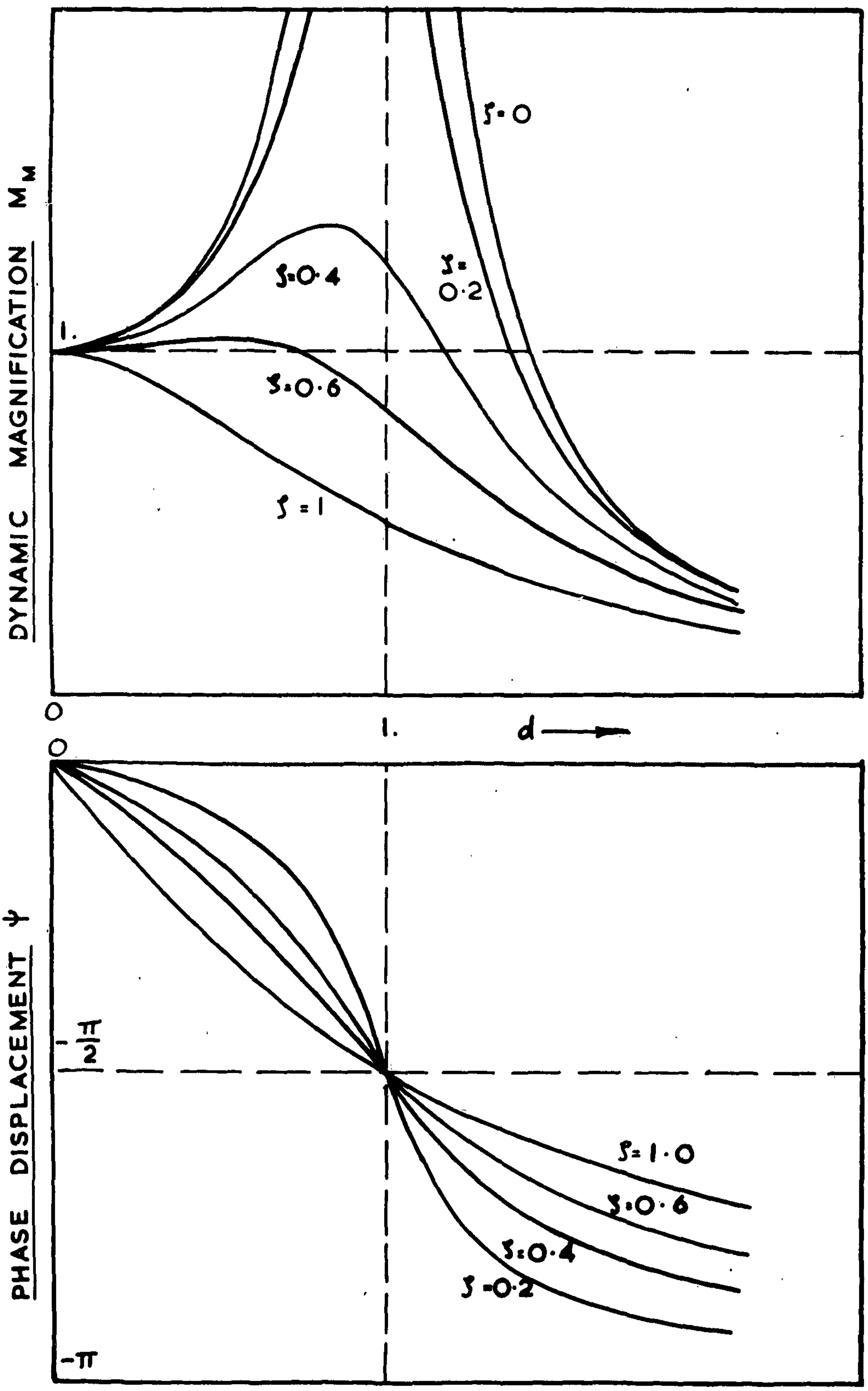
RESPONSE TO STEP INPUT VELOCITY ω_i.

FIG. 9.

SK 26336



ROOTS PLOTTED IN THE 'm' PLANE.

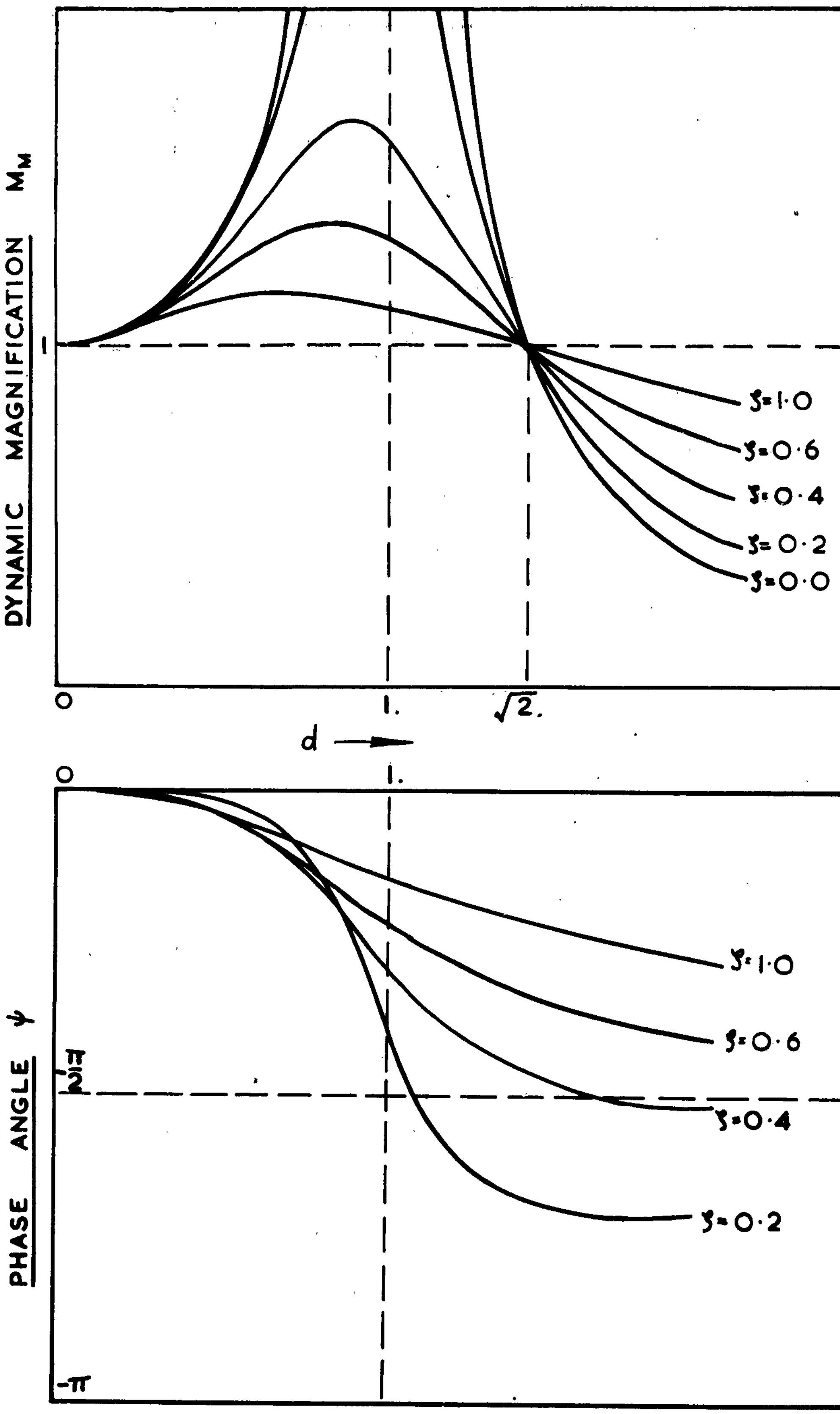


ZERO DISPLACEMENT ERROR SYSTEM

— HARMONIC INPUT.

SK 26338

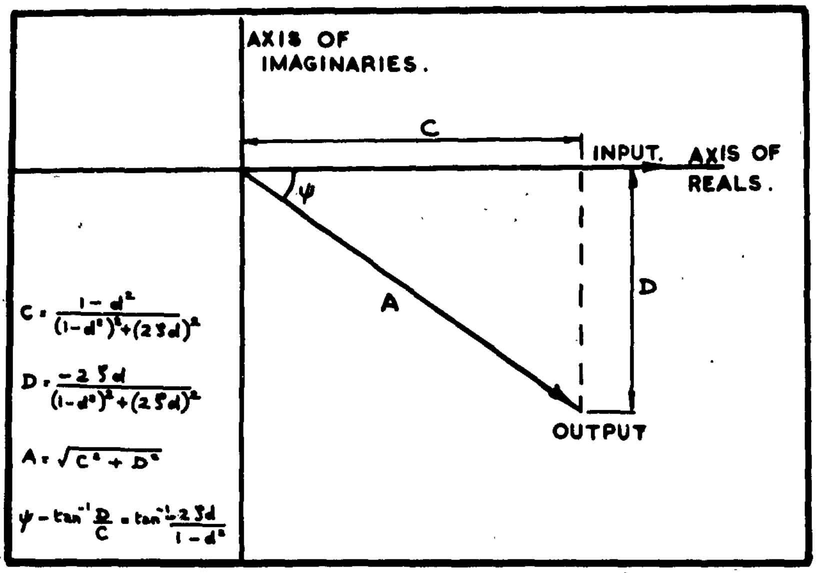
FIG. 11.



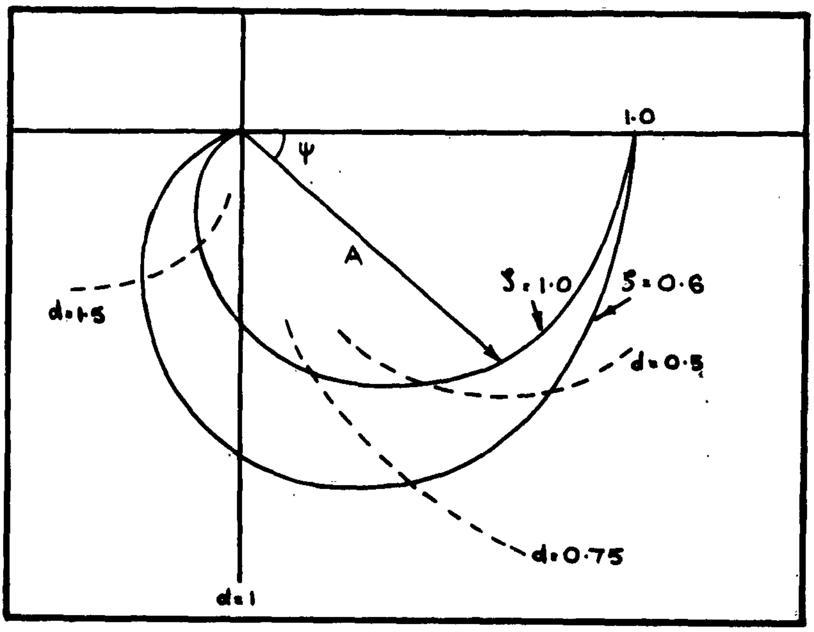
PROPORTIONAL PLUS DERIVATIVE OF ERROR CONTROL — HARMONIC INPUT.

FIG. 12.

SK 26339



VECTOR RELATIONSHIP.

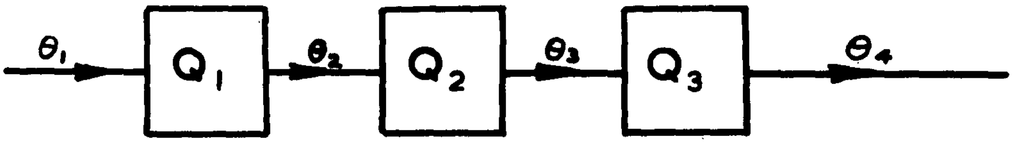


VECTOR LOCUS.

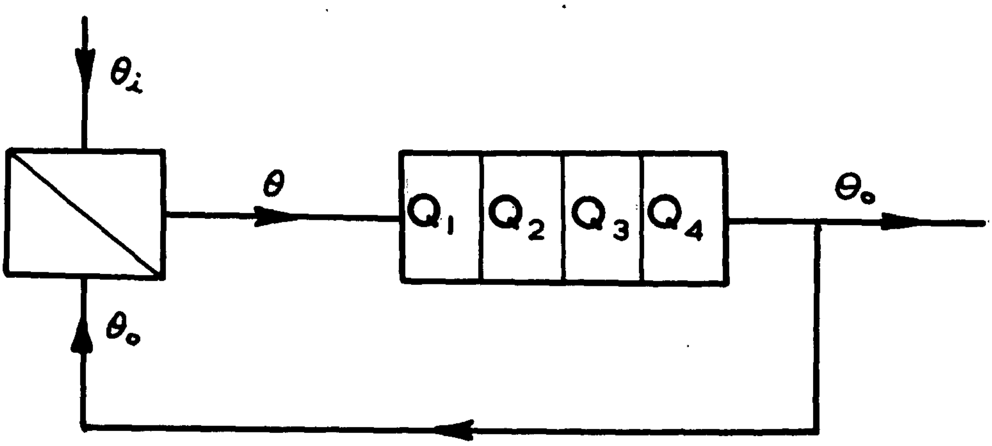
VECTOR LOCUS OF $\frac{\theta_o}{\theta_i}(jd) = Ae^{j\psi}$

SK 26340

FIG. 13.



OPEN CHAIN BLOCK DIAGRAM
USING TRANSFER FUNCTIONS.

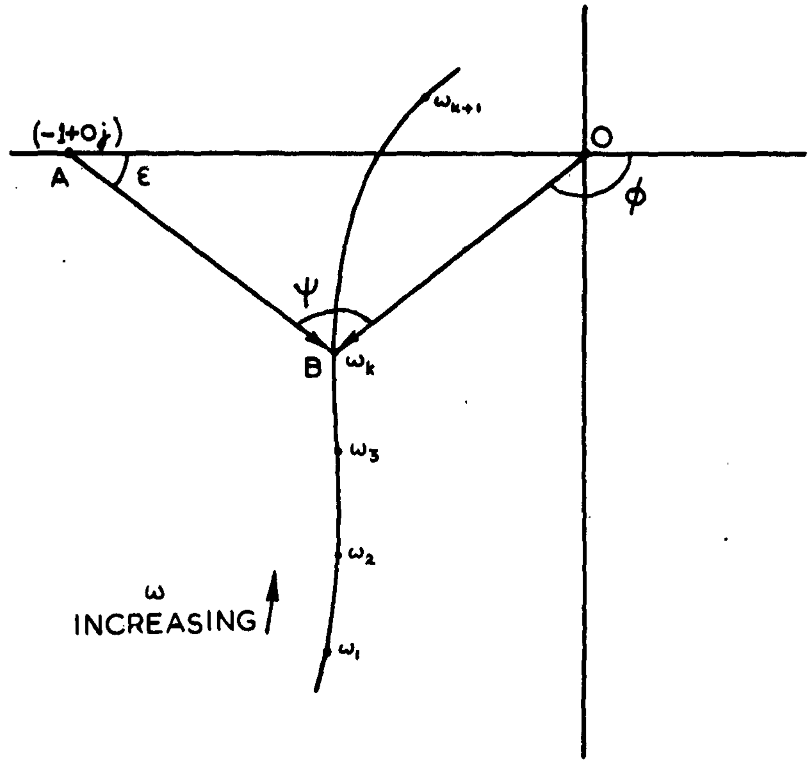


CLOSED LOOP SYSTEM BLOCK DIAGRAM
FOR UNITY FEEDBACK USING TRANSFER
FUNCTIONS.

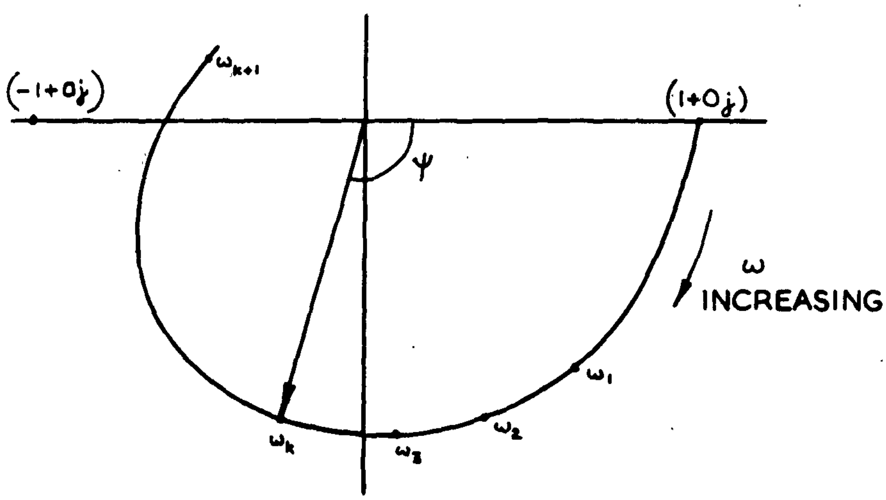
BLOCK DIAGRAMS AND
TRANSFER FUNCTIONS.

FIG. 14.

SK 26341



LOCUS OF KG(jω) OR NYQUIST DIAGRAM.

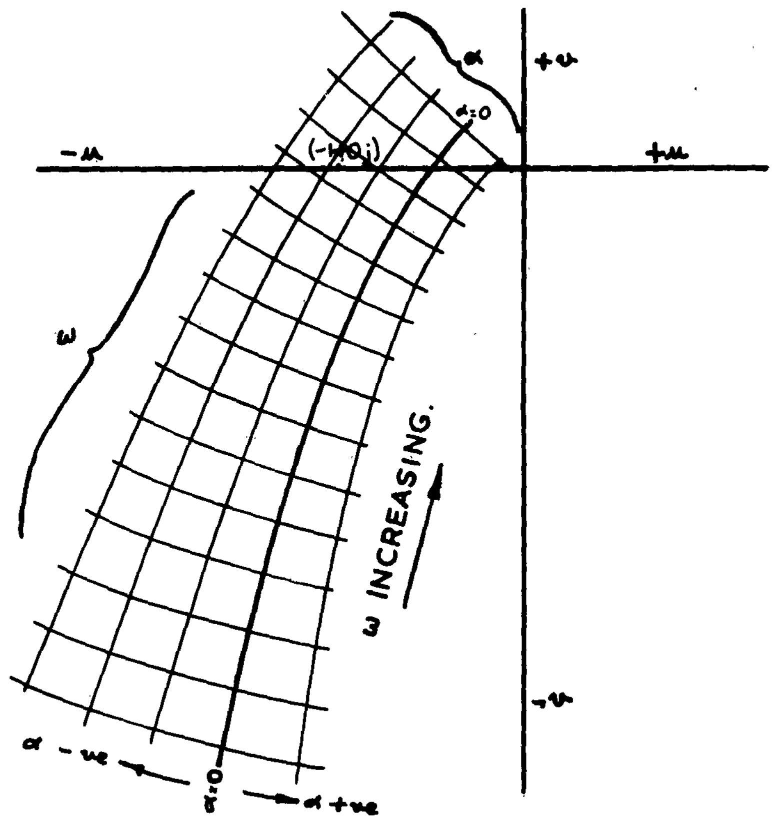


LOCUS OF $\frac{\theta_o}{\theta_i}(j\omega) = \frac{KG(j\omega)}{1 + KG(j\omega)}$

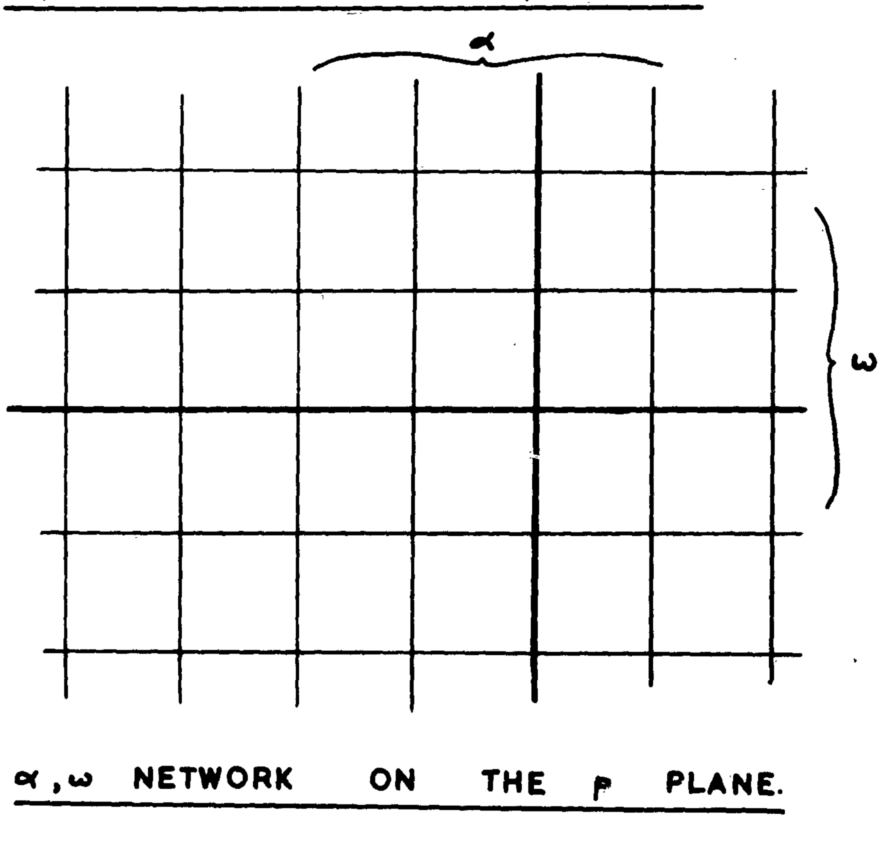
THE NYQUIST DIAGRAM AND GRAPHICAL DERIVATION OF $\frac{\theta_o}{\theta_i}(j\omega)$

FIG. 15.

SK 26342

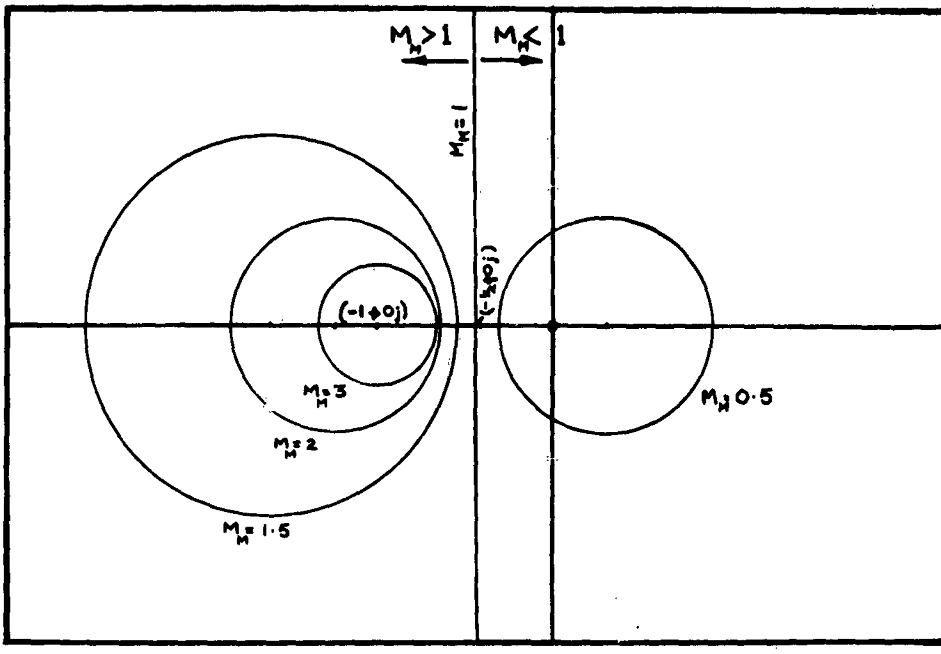


α, ω NETWORK ON THE $f(p)$ PLANE.

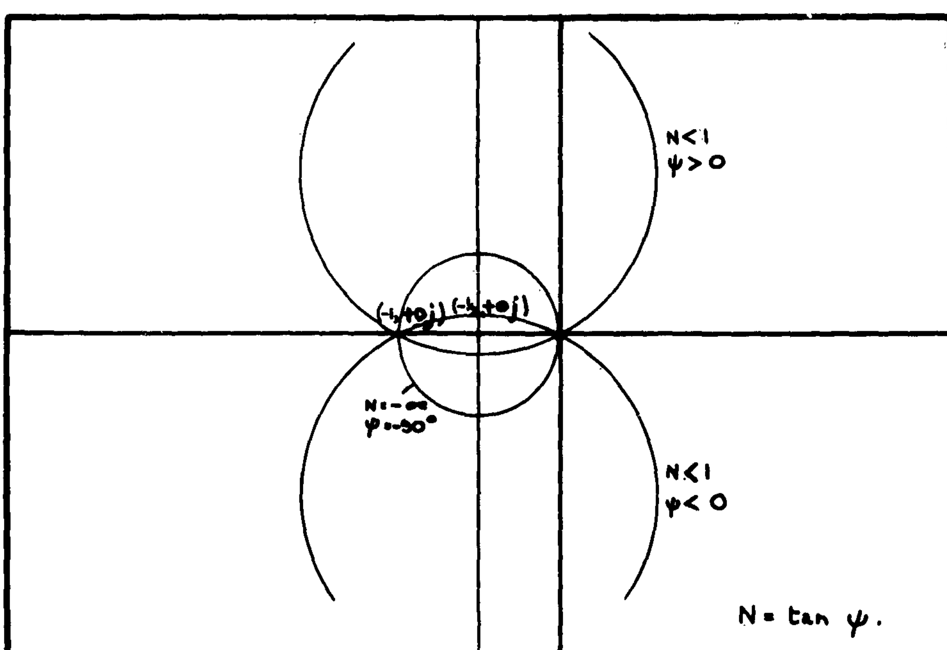


α, ω NETWORK ON THE p PLANE.

α, ω NETWORKS ON THE COMPLEX PLANE



CONSTANT M_M CONTOURS IN THE KG PLANE

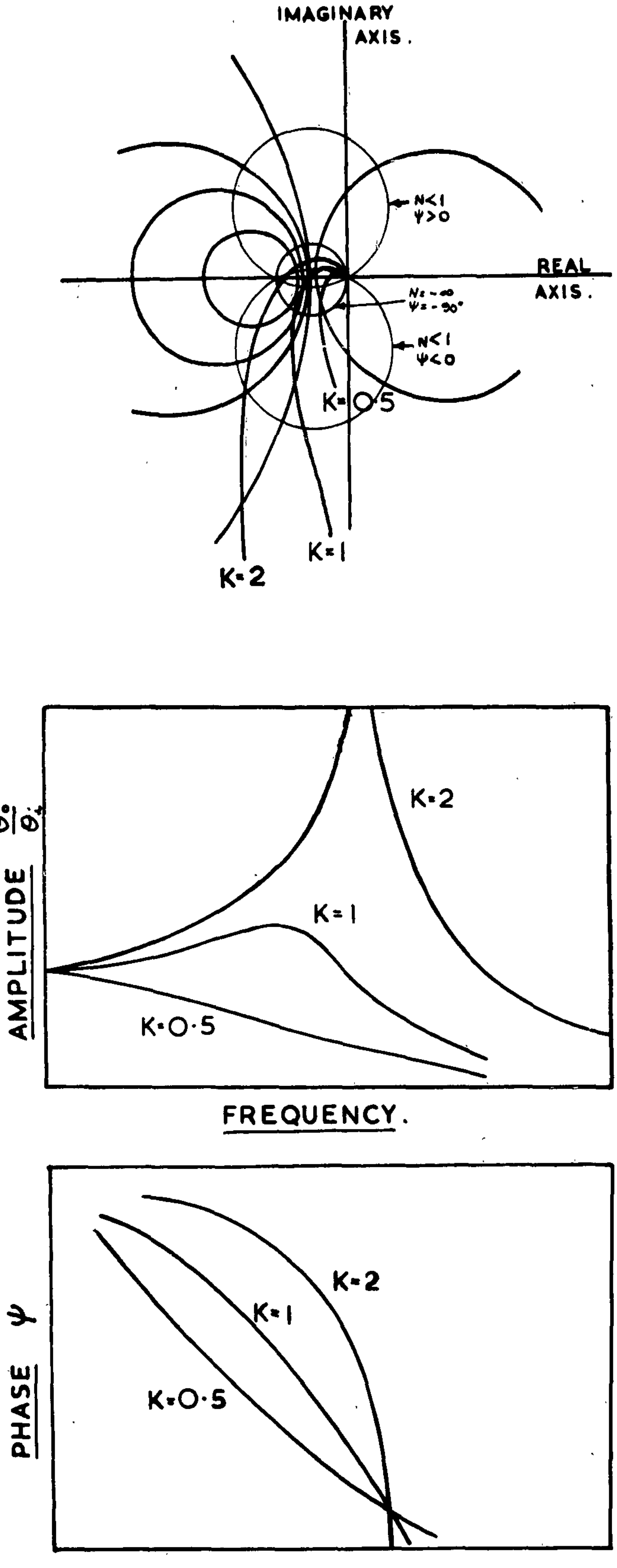


CONSTANT ψ CONTOURS IN THE KG PLANE.

CONSTANT MAGNIFICATION AND PHASE CONTOURS IN THE KG PLANE.

SK 26344

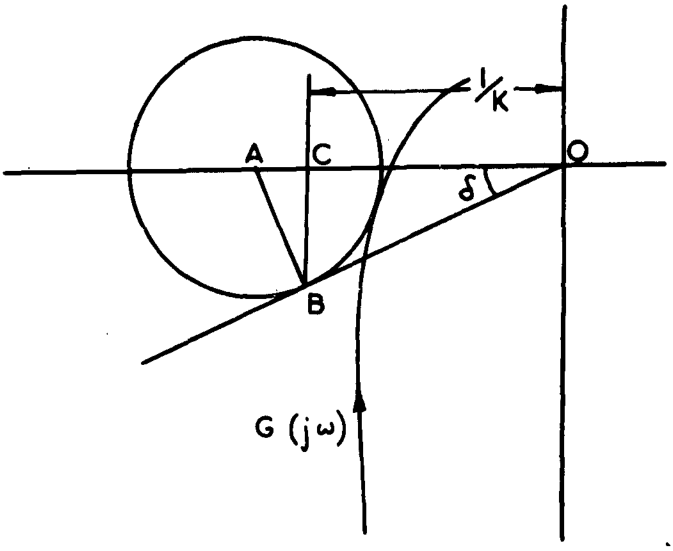
FIG. 17.



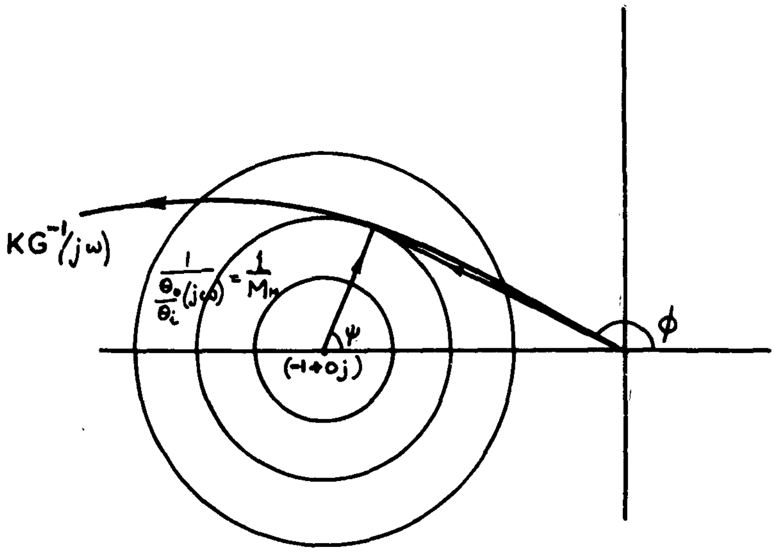
CHANGE IN K AND CROSS PLOT TO GIVE CONVENTIONAL INPUT OUTPUT AMPLITUDE AND PHASE CURVES AGAINST FREQUENCY.

FIG. 18.

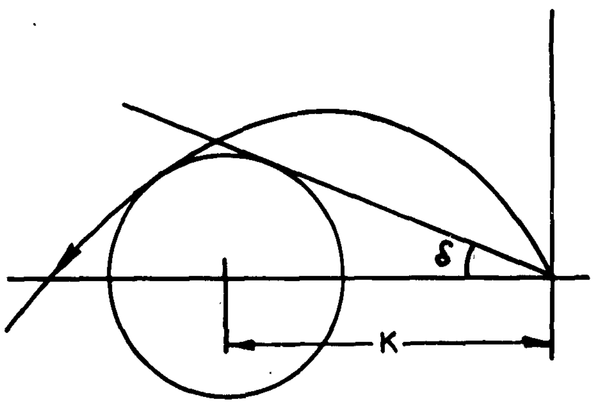
SK 26345



LOCUS OF G(jω)



INVERSE LOCUS OF $KG^{-1}(j\omega)$

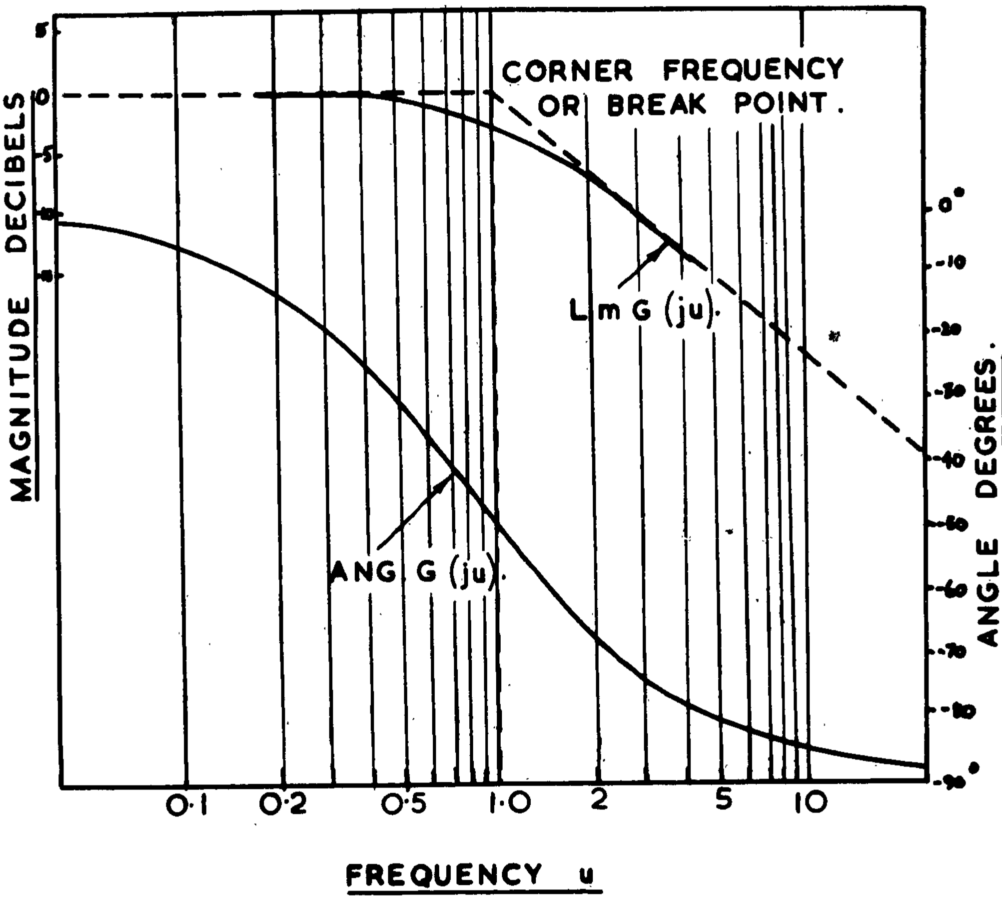


INVERSE LOCUS OF $G^{-1}(j\omega)$

LOCUS OF G(jω) AND INVERSE LOCI OF $KG^{-1}(j\omega)$ AND $G^{-1}(j\omega)$

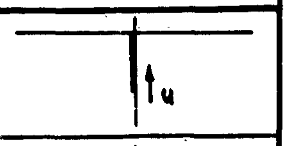
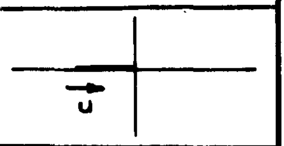
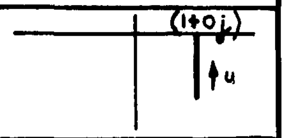
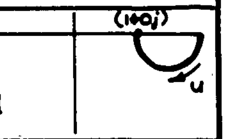
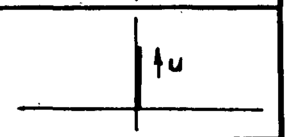
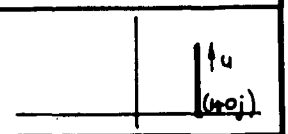
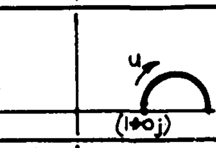
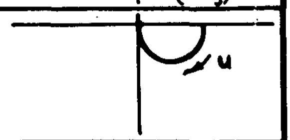
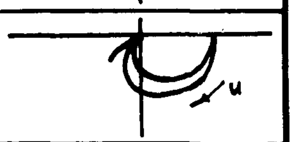
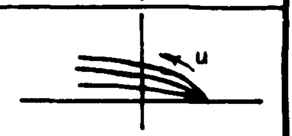
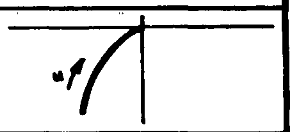
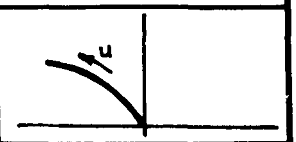
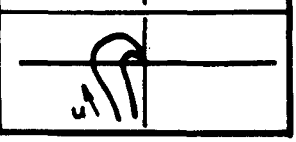
SK 26346

FIG. 19.



ASYMPTOTIC OR LOGARITHMIC FORM OF LOCUS.

SK 26347

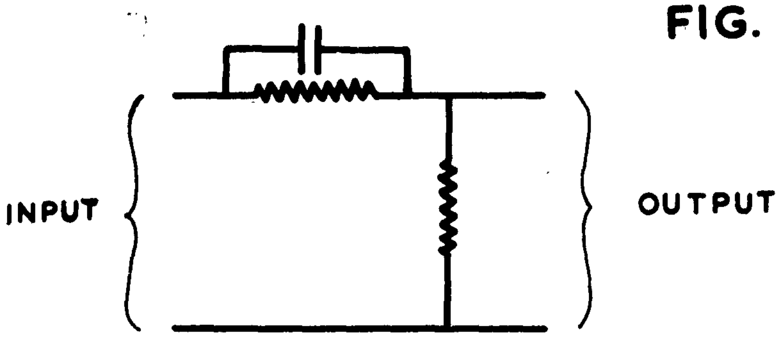
NO	DESCRIPTION.	LOCUS FOR FUNCTION $G(ju)$	LOCUS FOR FUNCTION $G^{-1}(ju)$	LOCUS SHAPE.
1	IDEAL FIRST ORDER OR INTEGRAL TERM.	$\frac{1}{ju}$	ju	
2	IDEAL SECOND ORDER OR DOUBLE INTEGRAL TERM.	$\frac{1}{(ju)^2}$	$(ju)^2$	
3	IDEAL PROPORTIONAL + INTEGRAL TERM.	$1+\frac{1}{ju}$	$\frac{ju}{1+ju}$	
4	PRACTICAL INTEGRAL TERM.	$\frac{1+ju}{1+\lambda ju}$	$\frac{1+\lambda ju}{1+ju}$	$\lambda>1$ 
5	IDEAL* DERIVATIVE TERM.	ju	$\frac{1}{ju}$	
6	IDEAL PROPORTIONAL + DERIVATIVE TERM.	$1+ju$	$\frac{1}{1+ju}$	
7	PRACTICAL DERIVATIVE TERM.	$\frac{1+\lambda ju}{1+ju}$	$\frac{1+ju}{1+\lambda ju}$	$\lambda>1$ 
8	SIMPLE EXPONENTIAL LAG.	$\frac{1}{1+ju}$	$1+ju$	
9	COMPLEX EXPONENTIAL LAG.	$\frac{1}{[(ju)^2+2\zeta ju+1]}$	$[(ju)^2+2\zeta ju+1]$	
10	INVERSE OF FUNCTION 9.	$[(ju)^2+2\zeta ju+1]$	$\frac{1}{[(ju)^2+2\zeta ju+1]}$	
11	FIRST ORDER + SIMPLE EXPONENTIAL LAG.	$\frac{1}{ju(1+ju)}$	$ju(1+ju)$	
12	INVERSE OF FUNCTION 11.	$ju(1+ju)$	$\frac{1}{ju(1+ju)}$	
13	FIRST ORDER + COMPLEX EXPONENTIAL LAG.	$\frac{1}{ju[(ju)^2+2\zeta ju+1]}$	$ju[(ju)^2+2\zeta ju+1]$	

TYPICAL LOCI SHAPES FOR $G(ju)$ AND $G^{-1}(ju)$.

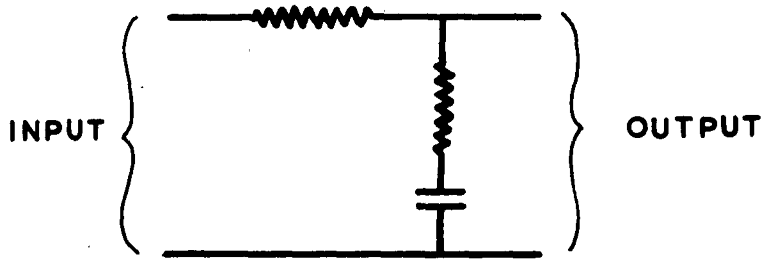
FIG. 20.

SK 26348

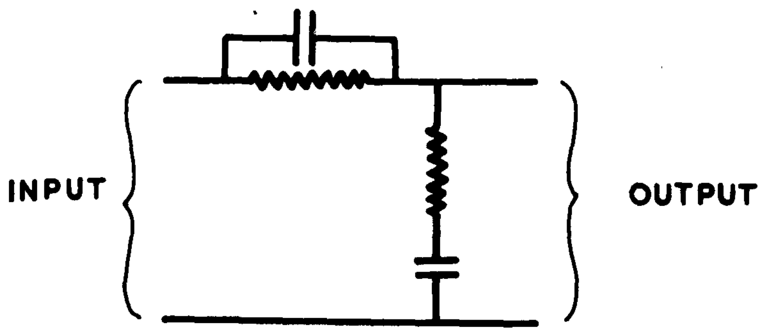
FIG. 21.



PROPORTIONAL + DERIVATIVE.



PROPORTIONAL + INTEGRAL.

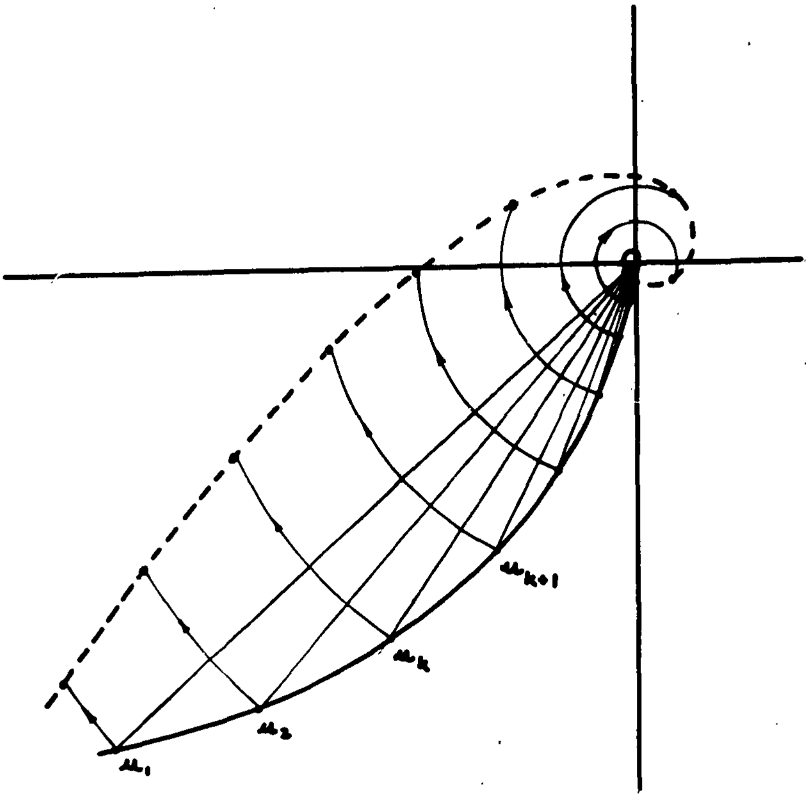


PROPORTIONAL + DERIVATIVE + INTEGRAL.

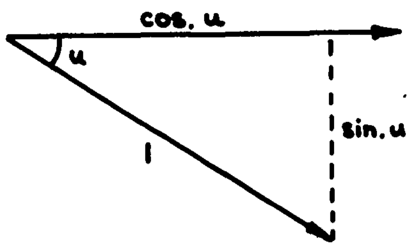
PASSIVE SHAPING CIRCUITS.

FIG.22.

SK 26350.



LOCUS OF $G(ju)$

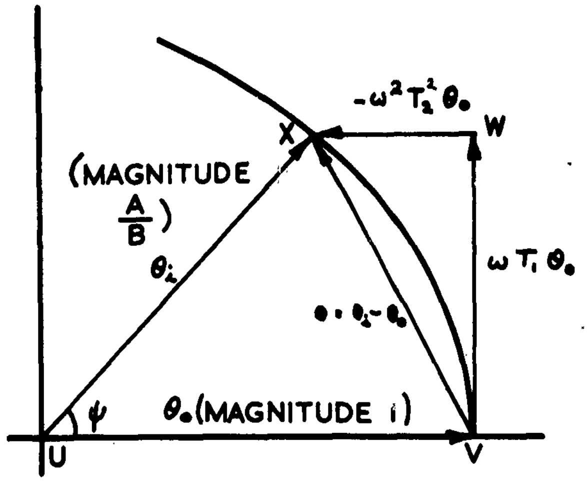


OPERATOR e^{-ju}

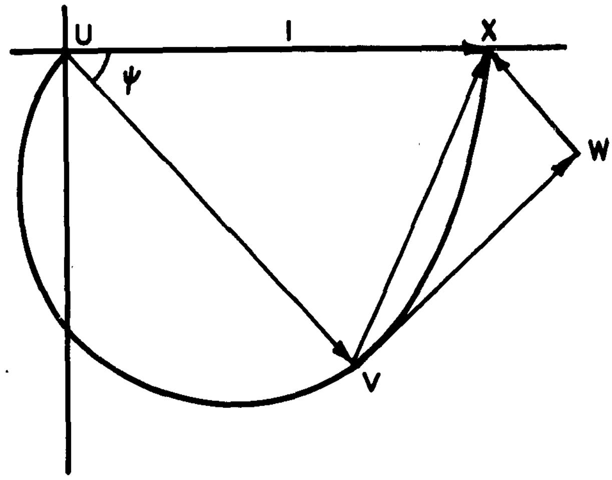
EFFECT OF A FINITE DELAY ON $G(ju)$

SK 26480.

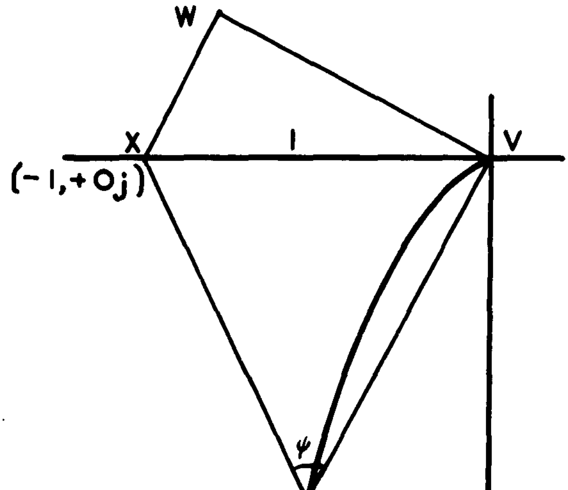
FIG. 23



INVERSE LOCUS. (a)



HARMONIC LOCUS. (b)



NYQUIST DIAGRAM. (c)

RELATIONSHIP BETWEEN THE THREE BASIC LOCI.

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As the flight conditions of gas turbine engined aircraft become more severe, engine control system complexity will be increased and the performance demanded of the control system in respect of the number of parameters to be controlled, the range of control, and the rate of response and sensitivity, will become more difficult to achieve.

Since the engine element itself is a dynamic system, the response of which is a function of basic engine performance and operating conditions, the study of engine response behaviour under controlled conditions must

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Since the engine element itself is a dynamic system, the response of which is a function of basic engine performance and operating conditions, the study of engine response behaviour under controlled conditions must

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result in the theory and technique of control system and engine performance becoming closely related.

In Part I of this paper an introduction is given to the fundamentals of control system theory relevant to the problem; this is intended as a guide to the non-specialist rather than as a treatise on servo-system.

The theory and technique of servo-system design are highly developed and provided the engine problem can be rationalised by acceptable means, the wealth of mathematical technique already available should enable the various problems to be resolved by established methods.

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